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Nets of Quadrics, and Theta-Characteristics of Singular Curves

C. T. C. Wall

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NETS OF QUADRICS, AND THETA-CHARACTERISTICS OF SINGULAR CURVES

By C. T. C. WALL, F.R.S.

Department of Pure Mathematics, University of Liverpool, P.O. Box 147, *Liverpool, U.K.*

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CONTENTS

The major part of this paper is devoted to enumerating all the many types of nets of quadrics in $\mathbb{C}^4$. An introductory section puts the invariant theory in context, and gives a framework for the classification.

A quadric $\boldsymbol{x}^{\mathrm{T}}(\lambda A_0+\mu A_1+\nu A_2)\boldsymbol{x}$ of the net has dual equation $\boldsymbol{X}^{\mathrm{T}}\operatorname{adj}(\lambda A_0+\mu A_1+\nu A_2)\boldsymbol{X}=0$. The *adjugate system* is the system of curves in the (λ,μ,ν) plane given

by these equations. The set B of base points of this system on the curve $0 = \Delta \equiv \det(\lambda A_0 + \mu A_1 + \nu A_2)$, together with the curve Δ, give system to the enumeration.

In a final section of the paper the calculations are used to provide evidence for conjectures of the following type (generalizing results known when $\Delta = 0$ is non-singular): each net determines and is determined by a square root of the canonical bundle on the curve Γ obtained from Δ by blowing B up; the set of square roots is an affine space over $\mathbb{F}_2$, and those arising are the zeros of a certain quadratic map.

Introduction

In a previous paper (Wall 1977) I gave a complete classification of nets of conics. The corresponding results for quadrics are enormously more complicated. However the analysis for non-singular discriminant is classical (Dixon 1902), and acting on a suggestion from Michael Atiyah that this was the correct approach I found that my lists could be presented in a systematic manner which not only showed considerable regularity of pattern but also led me to conjecture certain extensions of the classical theory.

This is presented in this paper. I have not deferred publication to wait for a proof of the conjectures, partly because this demands not only a much closer analysis of higher singularities even to obtain a compete formulation, but also a better understanding of the classical theory of theta functions and periods than I at present possess; and partly because I am interested in the nets of quadrics for their own sake, and wish to present the associated geometry as of intrinsic value not solely derived from this tantalizing relation with a deeper theory.

The paper is divided into four sections. In the introductory §0 we introduce notation, put the original problem in context and apply the techniques of geometric invariant theory (Mumford 1965). The detailed geometrical classifications are presented in §1: this involves numerous case distinctions which we ameliorate as far as appears possible. In §2 we introduce the adjugate system, whose base points will serve as a systematic basis for the enumeration, and give a partly heuristic calculation of multiplicities which serves as a check that the listing is complete. The system B of adjugate base points lies among the multiple points of the discriminant curve Δ. In §3 we present evidence that the nets are related to bundles over the curve Γ defined by blowing up Δ along B. This is reasonably explicit and convincing in the case when Δ has only double points, but the number of cases with triple points arising is not large enough to illustrate all the complications that could arise.

The subject of nets of quadrics is of course an old one, with references going back over a century. We refer particularly to the papers of Hesse (1855 *a*, *b*) where the adjugate system was first used in connection with the enumeration of nets with a given discriminant (see also Sturm 1869). The most extensive work in more recent times was the series of papers by W. L. Edge (1936–43). Edge's main interest was the study of the geometry and invariant theory of the general net, but he does mention a number of special cases. The following quotation from the introduction to the penultimate of these papers seems particularly apposite here: 'It would be a long undertaking to consider all the different specializations of a net of quadrics, and the importance of some of these would be small and out of all proportion to the labour involved in investigating them thoroughly'. The present investigation is by no means thorough in the sense envisaged, and we propose stability as a criterion of importance.

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0. Linear systems of quadrics in $\mathbb{C}^n$

We adopt the following notation. Take $x = (x_1, \ldots, x_n)$ as coordinates in $\mathbb{C}^n$, and write the equation of a quadric as

$$S = \sum_{i,j=1}^{n} a_{ij} x_i x_j,$$

where $A = (a_{ij})$ is a symmetric matrix. If we have several such, we may distinguish by an index: S_r, a^r_{ij}, $A^{(r)}$. A linear system of quadrics, of freedom k, may be written as $\sum_{r=0}^{k} \lambda_r S_r = 0$. One may seek to classify these forms under the action of the group $GL_{k+1}(\mathbb{C}) \times GL_n(\mathbb{C})$, where $GL_{k+1}(\mathbb{C})$ acts by linear substitutions of the λ_r and $GL_n(\mathbb{C})$ on the x_i. It is more convenient to modify this procedure, and first study the action of $GL_n(\mathbb{C})$.

If $k = 0$, the $GL_n(\mathbb{C})$-orbits are determined by the rank of S_0. If $k = 1$, we have a pencil of quadrics. The complete classification of pencils is well known (see for example, Jordan 1906; Gantmacher 1959; Hodge & Pedoe 1952). We assume this, also the notation for the classification introduced in the latter but originally due to Segre (1921). For a general pencil we can choose coordinates (i.e. operate by $GL_n(\mathbb{C})$) so that

$$S_0 = \sum_1^n x_i^2 \quad S_1 = \sum_1^n \alpha_i x_i^2.$$

The coordinates are then determined up to order and sign. Thus the corresponding isotropy group is finite.

To apply invariant theory, it is convenient to restrict the action to the subgroup $SL_n(\mathbb{C})$. The only invariant when $k = 0$ is clearly the determinant $\det(a_{ij})$. In general, if we write

$$\Delta = \Delta(\lambda) = \det\left(\sum_{r=0}^{k} \lambda_r a_{ij}^{(r)}\right)$$

then Δ is homogeneous of degree n in $\lambda_0, \ldots, \lambda_k$ and the coefficients of Δ are all invariants. We refer ambiguously to Δ, or to the locus $\Delta = 0$ considered as a hypersurface in $P_k(\mathbb{C})$, as the discriminant of the linear system.

If the discriminant locus has no repeated component, we can find a line in $P_k(\mathbb{C})$ meeting it in n distinct points. This line determines a subpencil of the linear system of the general type mentioned above. Hence the isotropy group (of the pencil, hence also of the linear system) is finite.

Now consider the case $k = 2$. Here, there are $\frac{3}{2}n(n+1)$ coefficients $a_{ij}^{(r)}$; $SL_n(\mathbb{C})$ has dimension n^2-1 and 'most' isotropy groups have dimension 0. The quotient space (in the sense of Rosenlicht 1963) is thus a union of manifolds with greatest dimension $\frac{1}{2}(n^2+3n+2)$. But this is precisely the number of coefficients in Δ. Since (Dixon 1902) a general plane n-ic curve Δ does correspond to a net, we deduce that for almost all Δ, there are a finite number of corresponding $SL_n(\mathbb{C})$-orbits of linear systems.

If the subpencil where only λ_0 and λ_1 are non-zero is general, and we normalize coordinates as above, it follows for most (we will shortly see, for all) homogeneous n-ics $\Delta(\lambda_1, \lambda_1, \lambda_2)$ taking the given values when $\lambda_2 = 0$ that the quadric S_2 is determined up to a finite ambiguity. For $k > 2$, we deduce again that (the SL_n-orbit of) the linear system is determined by Δ up to a finite ambiguity, but Δ will no longer be an arbitrary n-ic form. In the language of invariant theory, we deduce that the field of invariants will be a finite extension of the field $\mathbb{C}(\Delta)$ defined by the coefficients of Δ. That this extension is in general non-trivial we observed already in the paper on nets of conics.

For further precision we turn to the work of (Mumford 1965) on stability, according to which the system of invariants determines stable orbits and yields useful information in the semistable case. A linear system $\sum_0^k \lambda_r S_r$ *fails* to be stable (resp. semistable) if $SL_n(\mathbb{C})$ has a semisimple 1-parameter subgroup $\theta(t)$, depending on a multiplicative parameter t, such that each $\theta(t)\,S_r$ is bounded as $t\to\infty$ (resp. tends to 0 as $t\to\infty$).

Theorem 0.1. (i) *The system* $\sum_0^k \lambda_r S_r$ *is unstable if and only if, for some choice of coordinates x and some integer s,* $1 \leqslant s \leqslant \frac{1}{2}n$, *we have* $a_{ij}^{(r)} = 0$ *whenever* $0 \leqslant r \leqslant k$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant n-s$.

(ii) *It fails to be semistable if and only if for some coordinates x and some s,* $1 \leqslant s \leqslant \frac{1}{2}(n+1)$, *we have* $a_{ij}^{(r)} = 0$ *whenever* $0 \leqslant r \leqslant k$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant n+1-s$.

Proof. (i) As $\theta(t)$ is semisimple, it can be diagonalized: say as diag $(t^{\alpha_1}, \ldots, t^{\alpha_n})$ with $\alpha_1 \geqslant \ldots \geqslant \alpha_n$ and $\sum_1^n \alpha_i = 0$ (since we are in $SL_n(\mathbb{C})$), and not all α_i zero.

If for each i with $1 \leqslant i \leqslant n-1$ we have $\alpha_i + \alpha_{n-i} \leqslant 0$, then, summing, $2\sum_1^{n-1}\alpha_i \leqslant 0$. Thus $\alpha_n \geqslant 0$, and each $\alpha_i \geqslant \alpha_n \geqslant 0$. We deduce each $\alpha_i = 0$: a contradiction. Thus some $\alpha_s + \alpha_{n-s} > 0$; interchanging s and $n-s$ if necessary we may suppose $1 \leqslant s \leqslant \frac{1}{2}n$. But now as α_i is decreasing, $\alpha_i + \alpha_j > 0$ whenever $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant n-s$. Thus if each $\theta(t)\,S_r = \sum_{i,j=1}^n t^{\alpha_i+\alpha_j} a_{ij}^{(r)} x_i x_j$ is bounded as $t\to\infty$, we must have $a_{ij}^{(r)} = 0$ for all r whenever $i \leqslant s$, $j \leqslant n-s$.

Conversely if this condition is satisfied we define a 1-parameter subgroup as above by choosing

$$\alpha_i = \begin{cases} 1 & 1 \leqslant i \leqslant s \\ 0 & s+1 \leqslant i \leqslant n-s \\ -1 & n-s+1 \leqslant i \leqslant n \end{cases}$$

(note that $\Sigma\alpha_i = 0$). For this subgroup, each $\theta(t)\,S_r$ is indeed bounded as $t\to\infty$.

(ii) This is very similar. Normalize the subgroup as above. If each $\alpha_i + \alpha_{n+1-i} < 0$ $(1 \leqslant i \leqslant n)$, summing yields $\sum_1^n \alpha_i < 0$: a contradiction. We may thus suppose $\alpha_s + \alpha_{n+1-s} \geqslant 0$ and $1 \leqslant s \leqslant \frac{1}{2}(n+1)$. Now $\alpha_i + \alpha_j \geqslant 0$ whenever $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant n+1-s$; thus if each $\theta(t)\,S_r \to 0$ as $t\to\infty$ we must have $a_{ij}^{(r)} = 0$ for all r whenever $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant n+1-s$.

Conversely, if this holds consider the subgroup defined by

$$\alpha_i = \begin{cases} n-s & 1 \leqslant i \leqslant s \\ -1 & s+1 \leqslant i \leqslant n+1-s \\ s-n-1 & n+2-s \leqslant i \leqslant n: \end{cases}$$

here $\Sigma\alpha_i = s(n-s) - (n+1-2s) - (n+1-s)(s-1) = 0$ and $\alpha_i + \alpha_j \geqslant 0$ only if $i \leqslant s$, $j \leqslant n+1-s$ or vice versa. Thus indeed each $\theta(t)\,S_r \to 0$ as $t\to\infty$.

Our conclusion may be recast in more geometrical terms. We know the system fails to be semistable if and only if all invariants vanish (Mumford 1965) if and only if Δ vanishes identically (the field of invariants is a finite extension) if and only if the net consists of singular quadrics.

Corollary 1. *For any linear system of singular quadrics in* $P_{n-1}(\mathbb{C})$ *we can find subspaces* $P_{s-1}(\mathbb{C}) \subset P_{n-s}(\mathbb{C})$ *such that for each quadric S of the system,* $P_{s-1}(\mathbb{C})$ *lies in the vertex of* $S \cap P_{n-s}(\mathbb{C})$.

Conversely, one sees at once that any such quadric S is singular.

As to the case when the system is semistable but not stable, we have a similar conclusion with a flag $P_{s-1}(\mathbb{C}) \subset P_{n-s-1}(\mathbb{C})$. This is equivalent to saying that these subspaces are mutual polars with respect to all (non-singular) quadrics of the system. Here, we observe also that the matrices are now partitioned into blocks

$$\begin{array}{c} \quad\quad s \quad\; n-2s \quad s \\ A = \begin{bmatrix} 0 & 0 & B \\ 0 & C & D \\ B^{\mathrm{T}} & D^{\mathrm{T}} & E \end{bmatrix} \begin{array}{l} s \text{ rows} \\ (n-2s) \text{ rows} \\ s \text{ rows} \end{array} \end{array}$$

so that $\det A = (-1)^s (\det B)^2 \det C$. Hence Δ has a repeated factor of positive degree, and the locus $\Delta = 0$ has a repeated component.

Corollary 2. *For a linear system which is semistable but not stable, Δ is non-zero but has a repeated factor of positive degree.*

The converse assertion, though true for pencils and for nets of conics, admits a few exceptions when we come to nets of quadrics, as we shall see.

1. Enumeration of nets of quadrics in $\mathbb{C}^4$

1.0. *Preliminaries*

We turn to detailed consideration of the case mentioned. It will be convenient to modify notation slightly, and write the net as

$$\lambda S_0 + \mu S_1 + \nu S_2,$$

where $$S_0 = \sum_{i,j=1}^{4} a_{ij} x_i x_j, \quad S_1 = \sum_{i,j=1}^{4} b_{ij} x_i x_j \quad \text{and} \quad S_2 = \sum_{i,j=1}^{4} c_{ij} x_i x_j.$$

We usually adhere to this notation for the rest of the paper. We also write X_1 for the point $x_2 = x_3 = x_4 = 0$, etc. and L, M, N for the vertices of the triangle of reference $\lambda = 0$, $\mu = 0$, $\nu = 0$.

Our objective is, for each possible $\Delta = \Delta(\lambda, \mu, \nu)$ to enumerate the $SL_n(\mathbb{C})$-orbits of nets with discriminant Δ. We omit the phrase '$SL_n(\mathbb{C})$-orbits of', and speak of counting nets. It is thus necessary to normalize the coordinates (x_1, x_2, x_3, x_4) rather carefully in each case.

We arrange the possible Δ as follows. Leave unstable nets to the end; next to last we deal with nets with a singular subpencil. The rest we classify as follows: ordinary singularities only; higher double point; triple point. Further classification is treated in each section as it occurs.

1.1. *Nets of conics*

The results of § 0 will only be needed in the following form. A net of quadrics in $\mathbb{C}^4$ fails to be stable if and only if coordinates can be chosen so that the matrix of each of S_0, S_1 and S_2 has the form (here 0 denotes a zero entry and * that no restriction is imposed):

$$\text{case } s = 1 \quad \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & * & * & * \\ 0 & * & * & * \\ * & * & * & * \end{bmatrix}; \quad \text{case } s = 2 \quad \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}.$$

In case $s = 1$, each quadric of the net meets $x_4 = 0$ in a line-pair with double point at X_1; in case $s = 2$, each quadric contains the line $x_3 = x_4 = 0$.

Similarly, a net fails to be semistable if and only if we can reduce to matrices of forms

$$\text{case } s=1 \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}; \quad \text{case } s=2 \quad \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ * & * & * & * \end{bmatrix}.$$

In case $s = 1$, each quadric of the net is a cone with X_1 as vertex; in case $s = 2$, each quadric meets $x_4 = 0$ in the repeated line X_1X_2.

For the case of nets of conics, the corresponding forms are,

$$\text{not stable:} \quad \begin{bmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{bmatrix},$$

$$\text{not semistable:} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix}.$$

Semistable nets

Here we must have $s = 1$ in theorem 0.1. If the terms in the diagonal (i.e. in positions (1, 3) and (2, 2)) are proportional, we may take each as λ. This yields a unique net $\lambda(x_2^2+2x_1x_3)+\mu(2x_2x_3)+\nu x_3^2$ (type H). Otherwise we have, say,

$$\begin{bmatrix} 0 & 0 & \nu \\ 0 & \mu & a_{23}\lambda+b_{23}\mu+c_{23}\nu \\ \nu & a_{23}\lambda+b_{23}\mu+c_{23}\nu & a_{33}\lambda+b_{33}\mu+c_{33}\nu \end{bmatrix}$$

and may change coordinates by $x_1' = x_1+\alpha x_2+\beta x_3$, $x_2' = x_2+\gamma x_3$. We choose α and β, to make c_{23} and c_{33} vanish. If $a_{23} = 0$, choose γ to make b_{23} vanish and set $\lambda' = a_{33}\lambda+b_{33}\mu$ yielding $\lambda x_3^2+\mu x_2^2+2\nu x_1x_3$ (type G). If $a_{23} \neq 0$, choose to γ to make a_{33} vanish and set $\lambda' = a_{23}\lambda+b_{23}\mu$. Multiplying all coordinates by suitable constants we can adjust to $b_{33} = 1$ (type F) or $b_{33} = 0$ (type G*).

Stable nets

Since there is no common base point with a common tangent for the whole net, we can find a subpencil with no repeated base point, and so meeting Δ in distinct points; thus Δ has no repeated component.

For irreducible Δ, we normalize

$$\Delta = -\mu^2\nu+\lambda^3+p\lambda\nu^2+q\nu^3 = -\mu^2\nu+f(\lambda,\nu) \quad \text{and} \quad S_0 = 2x_1x_3+x_2^2,\ S_1 = 2x_2x_3$$

and then find

$$S_2 = -x_1^2-2gx_2^2+cx_3^2+2gx_1x_3 \quad \text{and} \quad f(\lambda,\nu) = (\lambda-2g\nu)(\lambda^2+2g\lambda\nu+(c+g^2)\nu^2).$$

If Δ is irreducible, we single out one of the three roots of f; we have 3 nets (type A). If Δ has a node, f has a repeated root, we may choose it (type B) or the simple root (type B*). If Δ has a cusp, f is a perfect cube and we have just one net (type C). Enumeratively, we see that type B occurs with multiplicity 2, and C with multiplicity 3.

If Δ is reducible there is a singular subpencil. If this has type [;1;] we can take

$$S_0 = 2x_1 x_3, \ S_1 = 2x_2 x_3$$

and then

$$\Delta = -c_{11}\mu^2\nu + 2c_{12}\lambda\mu\nu - c_{22}\lambda^2\nu + 2(c_{12}c_{23} - c_{22}c_{13})\lambda\nu^2 + 2(c_{12}c_{13} - c_{11}c_{23})\mu\nu^2 + (\det C)\nu^3;$$

the coefficients of Δ yield (in turn) the values of c_{11}, c_{12}, c_{22}; linear equations for c_{13}, c_{23} with discriminant $c_{11}c_{22} - c_{12}^2$ and (if these are solved) a linear equation for c_{33} with coefficient $c_{11}c_{22} - c_{12}^2$. So if $\Delta = \nu\Delta_1$, and ν meets Δ_1 in two distinct points, there is a unique net; otherwise, we may take $c_{11} = c_{12} = 0$ and the net is unstable. There are essentially two cases here: Δ_1 irreducible (type D*) or a line-pair (type E*), and in the latter case if we take $\Delta_1 = 2\mu\nu$ we obtain $S_3 = 2x_1x_2$ showing that the rôles of the singular lines are symmetrical.

Otherwise the singular subpencil $\nu = 0$ has a common vertex – say X_1. If X_1 lies on S_2, the net is unstable, so we may take $x_1 = 0$ as the polar of X_1 with respect to S_2, and $c_{11} = 1$, $c_{12} = c_{13} = 0$. Then $\Delta = \nu\Delta_1$ where Δ_1 is the discriminant of the net (or pencil) cut on $x_1 = 0$. If this is a net, Δ_1 is irreducible and determines the remaining coefficients up to normalization of $x_2 : x_3$; ν may be a chord of Δ_1 (type D) or a tangent (type F*). If it is a pencil it must be of type [1, 1] (else our net is unstable), and ν may not go through the vertex (else our net reduces to a pencil). Thus we have a single type (E) and Δ is a triangle. We may take coordinates to reduce it to $\lambda x_1^2 + \mu x_2^2 + \nu x_3^2$.

Dr Hirschfeld has kindly supplied references to the original sources of this enumeration in the complex case (Jordan 1906) and in the real case (Campbell 1928).

1.2. *Δ with ordinary singularities; no singular pencil*

For irreducible Δ, it will suffice for our purposes to count the number δ of ordinary double points and κ of cusps. Plücker's formula gives $3 - \delta - \kappa$ for the genus, hence $\delta + \kappa \leqslant 3$: we have ten cases. We also have the case when Δ breaks up into two conics, meeting in four distinct points.

For the less degenerate cases, a direct enumeration seems forbiddingly complex. We will be able to predict the results with some confidence in § 2; here we confine ourselves to one argument which, though incomplete, gives a pattern which serves as a useful guideline in several cases below.

Lemma 1.1. *Suppose S_0 a plane-pair, so that* $\det(\lambda S_0 + \mu S_1)$ *has a repeated factor μ. Then the other two roots coincide if and only if S_1 meets (at least) one of the planes of S_0 in a degenerate conic.*

Proof. This can be extracted from the enumeration of pencils, or we may proceed directly as follows. Take coordinates with S_0 given by $2x_1x_2 = 0$; recall S_1 has matrix (b_{ij}), and we write B_{ij} for the determinant obtained by crossing out the ith row and jth column. Then

$$\det(\lambda S_0 + \mu S_1) = \mu^4 \det B - 2\lambda\mu^3 B_{12} - \lambda^2\mu^2(b_{33}b_{44} - b_{34}^2).$$

The condition for equal roots is $B_{12}^2 = -(b_{33}b_{44} - b_{34}^2)\det B$. But the right hand side coincides (by a standard determinantal identity) with $B_{12}^2 - B_{11}B_{22}$, so we have either $B_{11} = 0$ (and the intersection with $x_1 = 0$ is degenerate) or $B_{22} = 0$.

Now consider a net $\lambda S_0 + \mu S_1 + \nu S_2$ with S_0 a plane-pair. Then Δ has a double point at L, and the equation for tangents from L to Δ is obtained as above (with $\mu S_1 + \nu S_2$ replacing μS_1), so is a sextic in $\mu : \nu$ which splits as a product of two cubics, corresponding respectively to degenerate intersections with the two planes of S_0. If we are given Δ but not the net, we have the set of six tangents, and there are ten ways of partitioning these into two sets of three. We may thus expect precisely ten nets with discriminant Δ and L corresponding to a plane-pair; and if Δ has no further singularity this is indeed the case.

We shall find in more degenerate cases that an otherwise mysterious enumeration can be seen as a special form of this partitioning of the set of tangents.

Here, a further point comes up. If Δ has a further double point M, it is desirable to know whether the corresponding quadric of the net is a cone or a pair of planes. Provided the line LM is not a component of Δ, we find that it counts with multiplicity 2 among the tangents from L to Δ. Clearly if S_1 is a plane-pair it meets both planes of S_0 in degenerate conics, so LM belongs to each of the systems of three tangents from L to Δ. Conversely suppose LM belongs to both sets, and meets Δ again at M. If S_1 is a cone, as it meets both planes of S_0 in line-pairs its vertex must lie in each plane. But this point is then a common vertex of S_0 and S_1, hence of all quadrics of the pencil they define, contradicting our assumption that $\nu = 0$ is not a component of Δ. This proves

LEMMA 1.2. *A net with plane-pair corresponding to a double point* L *of Δ induces a partition of the set of tangents from* L *to Δ into two sets of three. A further double point* M, *such that* LM *is not a component of Δ, corresponds to a plane-pair if and only if* LM *belongs to each set.*

Now suppose Δ has three distinct double points.

These cannot be collinear (else their join would have to be a component of Δ): take them as L, M and N. The standard quadratic transformation $(\lambda, \mu, \nu) \to (1/\lambda, 1/\mu, 1/\nu)$ now takes Δ to a conic Σ: we may assume Σ passes through none of L, M and N (else Δ would have λ, μ or ν as a factor).

Δ is irreducible if and only if Σ is: it has an ordinary node at L unless Σ touches λ, when it has a cusp there (similarly for M, N). If Σ is a line-pair, Δ is a pair of conics: they meet in 4 distinct points unless the vertex of Σ lies on $\lambda\mu\nu = 0$ – say on $\lambda = 0$ – when the conics touch at L, and meet again in M, N. If Σ is a repeated line, Δ is a repeated conic. We shall include these special cases also.

As Δ has double points at L, M, N its equation has the form $A\mu^2\nu^2 + B\nu^2\lambda^2 + C\lambda^2\mu^2 + 2\lambda\mu\nu(F\lambda + G\mu + H\nu) = 0$. The matrix of Σ is then

$$\begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix}$$

We have the restrictions $A \neq 0$, $B \neq 0$, $C \neq 0$ and the conditions for cusps at L, M, N are respectively

$$BC = F^2, \quad CA = G^2, \quad AB = H^2.$$

The pencils $\nu = 0$, etc. have discriminant $c\lambda^2\mu^2$, where $c \neq 0$, hence have type [2, 2], [2, (1,1)] or [(1,1), (1,1)]: this is determined by whether vertices correspond to cones or to plane-pairs. We adopt this as our main principle of classification.

1.2.1. *Case of three plane-pairs*

We may take $S_0 = 2x_1x_2$, $S_1 = 2x_3x_4$ and then

$$S_2 = 2(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)(b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4).$$

Since the four planes of S_0, S_2 are in general position, $a_3b_4 \neq a_4b_3$: similarly $a_1b_2 \neq a_2b_1$. Coordinates are unique up to interchanging x_1 and x_2 (or x_3 and x_4) and

$$x_1' = \rho x_1, \quad x_2' = \rho^{-1}x_2, \quad x_3' = \sigma x_3, \quad x_4' = \sigma^{-1}x_4.$$

A direct calculation (note that the terms divisible by ν^3 are known to cancel) shows that Σ has matrix

$$\begin{bmatrix} (a_1 b_2 - a_2 b_1)^2 & H & a_1 b_2 + a_2 b_1 \\ H & (a_3 b_4 - a_4 b_3)^2 & a_3 b_4 + a_4 b_3 \\ a_1 b_2 + a_2 b_1 & a_3 b_4 + a_4 b_3 & 1 \end{bmatrix}$$

where

$$H = 2(a_1 b_2 + a_2 b_1)(a_3 b_4 + a_4 b_3) - (a_1 b_4 + a_4 b_1)(a_2 b_3 + a_3 b_2) - (a_1 b_3 + a_3 b_1)(a_2 b_4 + a_4 b_2)$$
$$= (a_1 b_2 + a_2 b_1)(a_3 b_4 + a_4 b_3) - 2(a_1 a_2 b_3 b_4 + b_1 b_2 a_3 a_4).$$

Thus $\det \Sigma = -4(a_1 a_2 b_3 b_4 - a_3 a_4 b_1 b_2)^2$. Moreover,

$$\text{L is a cusp} \Leftrightarrow a_3 a_4 b_3 b_4 = 0$$
$$\text{M is a cusp} \Leftrightarrow a_1 a_2 b_1 b_2 = 0,$$

and $$\text{N is a cusp} \Leftrightarrow (a_1 b_4 - a_4 b_1)(a_2 b_3 - a_3 b_2)(a_1 b_3 - a_3 b_1)(a_2 b_4 - a_4 b_2) = 0,$$

more geometrically, for L to be a cusp both planes of S_0, one of S_1 and one of S_2 must concur; similarly for the others.

For enumeration, we must be a little more careful. The matrix Σ determines $a_1 b_2 + a_2 b_1$, $a_3 b_4 + a_4 b_3$, $a_1 a_2 b_3 b_4 + a_3 a_4 b_1 b_2$ and, up to sign, $a_1 b_2 - a_2 b_1$ and $a_3 b_4 - a_4 b_3$. Thus we have the pairs $(a_1 b_2, a_2 b_1)$ and $(a_3 b_4, a_4 b_3)$: interchanging coordinates if necessary, this determines $a_1 b_2$, $a_2 b_1$, $a_3 b_4$ and $a_4 b_3$. Moreover, we know the product of all, hence can solve for the pair $(a_1 a_2 b_3 b_4, a_3 a_4 b_1 b_2)$. Interchanging the a_i with the b_i (if necessary), we thus know $a_1 a_2 b_3 b_4$ and $a_3 a_4 b_1 b_2$. As we can adjust (using coordinate changes) by ρ, σ; also use $a_i' = a_i \tau$, $b_i' = b_i \tau^{-1}$ we can in general normalize all these values by $b_2 = b_3 = b_4 = 1$, when all the rest are determined. Similar arguments cover the cases when certain a_i, b_j vanish; with the exception of the case when $a_1 b_2 = 0$, $a_3 b_4 = 0$ and $a_1 a_2 b_3 b_4 = a_3 a_4 b_1 b_2 = 0$ when there is insufficient data to determine whether $a_1 = a_3 = 0$ or $b_2 = b_4 = 0$. However, even here these cases are seen to be equivalent on interchanging x_1 with x_2, x_3 with x_4 and the *a*s with the *b*s. We thus obtain a unique net in all cases.

If there are no cusps (indeed, if L, M are not cusps) no coefficients vanish. If L is a cusp, $a_3 a_4 b_3 b_4 = 0$. The various cases being equivalent, suppose $a_3 = 0$. From $a_3 b_4 \neq a_4 b_3$ follows that a_4 and b_3 are non-zero. If moreover $\det \Sigma \neq 0$ then $a_1 a_2 b_3 b_4 \neq 0$. Thus the only other coefficients that may vanish are b_1 and b_2: both cannot be 0 together, and one is if and only if M, too, is a cusp. Suppose $a_3 = b_1 = 0$: then N is a cusp if and only if $a_2 b_4 - a_4 b_2 = 0$. Thus each case does occur: uniquely, as was noted above.

In particular we have a stable net for the case when Δ is a repeated conic. We may verify that we have the net of quadrics through a twisted cubic curve.

Consider finally what happens to the fourth node in the singular case. Here $a_1 a_2 b_3 b_4 = a_3 a_4 b_1 b_2$; the vertex of Σ is the intersection of

$$\nu + (a_3 b_4 + a_4 b_3)\mu + (a_1 b_2 + a_2 b_1)\lambda = 0 \quad \text{and} \quad a_1 a_2 \lambda + a_3 a_4 \mu = 0$$

(the same as $b_1 b_2 \lambda + b_3 b_4 \mu = 0$). As this does not lie on $\lambda\mu\nu = 0$, the a_i and b_i are all non-zero. Now

$$S_0 = 2y_0^+ y_0^-, \; S = 2y_1^+ y_1^-, \; S_2 = 2y_2^+ y_2^-,$$

where
$$y_0^+ = x_1, \quad y_1^+ = x_3, \quad y_2^+ = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$$
$$y_0^- = x_2, \quad y_1^- = x_4, \quad y_2^- = b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4.$$

Set
$$y_3^- = a_1 a_3^{-1} b_3 x_1 + b_2 x_2 + b_3 x_3 + a_2^{-1} a_4 b_2 x_4,$$
$$y_3^+ = a_3 b_1 b_3^{-1} x_1 + a_2 x_2 + a_3 x_3 + a_2 b_2^{-1} b_4 x_4.$$

Since $a_1 a_2 b_3 b_4 = a_3 a_4 b_1 b_2$, we see that $y_0^{\epsilon_0}, y_1^{\epsilon_1}, y_2^{\epsilon_2}$ and $y_3^{\epsilon_3}$ are concurrent whenever $\epsilon_0 \epsilon_1 \epsilon_2 \epsilon_3 = -1$: we have a pair of desmic tetrahedra. The vertex $\left(\frac{-1}{a_1 a_2}, \frac{1}{a_3 a_4}, \phi = \frac{b_1}{a_1} + \frac{b_2}{a_2} - \frac{b_3}{a_3} - \frac{b_4}{a_4}\right)$ of Σ yields a node $(-a_1 a_2, a_3 a_4, \phi^{-1})$ of Δ corresponding to $S' = S_2 + \phi(a_3 a_4 S_1 - a_1 a_2 S_0)$. It is now easily seen that $2 y_3^+ y_3^- = S'$.

1.2.2. *Case of two plane-pairs and a cone*

Let L correspond to the cone: take $S_0 = x_1^2 + 2x_2 x_3$. Then (cf. lemma 1.1) each of S_1, S_2 is compounded of a plane not through the vertex and a non-tangent plane through the vertex. We may thus take
$$S_1 = 2x_1(a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4)$$
$$S_2 = 2x_4(b_1 x_1 + b_2 x_2 + b_3 x_3)$$
where $a_4 \neq 0$, $b_1^2 + 2b_2 b_3 \neq 0$ and, as the four planes are in general position, $a_2 b_3 \neq a_3 b_2$. We find
$$\Sigma = \begin{bmatrix} (a_2 b_3 - a_3 b_2)^2 & 2a_1 b_2 b_3 - b_1(a_2 b_3 + a_3 b_2) & -a_4(a_2 b_3 + a_3 b_2) \\ 2a_1 b_2 b_3 - b_1(a_2 b_3 + a_3 b_2) & b_1^2 + 2b_2 b_3 & a_3 b_1 \\ -a_4(a_2 b_3 + a_3 b_2) & a_4 b_1 & a_4^2 \end{bmatrix},$$
with $\det \Sigma = -4a_4 b_2^2 b_3^2(a_1^2 + 2a_2 a_3)$. However, if (for example) $b_2 = 0$, the line $x_1 = x_3 = 0$ is common to all quadrics of the net, which is thus unstable. Hence $a_4 b_2^2 b_3^2 \neq 0$, and Σ is non-singular unless $a_1^2 + 2a_2 a_3 = 0$. Since b_2, b_3 and a_4 may not vanish, there cannot be a cusp at L and the conditions for cusps at M, N reduce respectively to
$$a_2 a_3 = 0,$$
$$(a_2 b_3 - a_3 b_2)^2 = 2(a_1 b_2 - a_2 b_1)(a_1 b_3 - a_3 b_1).$$

Again our unsymmetrical choice of coordinates leads to apparent complications here: we have (say)
$$S_1 = 2x_1 y_1, \quad S_2 = 2x_5 y_5$$
where $y_5 = x_4$ does not go through the vertex of S_0; x_5 does. A unique linear combination $z = y_1 - a_4 y_5$ does pass through this vertex; M is a cusp if $z = x_1 = 0$ lies on S_0, and N is a cusp if $z = x_5 = 0$ does. The criterion for Σ to be singular is that $z = 0$ touches S_0. Thus projecting from X_4 we have the conic S_0; chords $x_1 = 0$, $x_5 = 0$ not meeting on X_4, and a further line $z = 0$ which may be tangent or pass through one or more of the intersections of $x_1 = 0$, $x_5 = 0$ with S_0.

We may still change coordinates by $x_2' = \rho x_2$, $x_3' = \rho^{-1} x_3$: we normalize ρ (up to sign) by requiring $b_2 = b_3$.

Given Σ, we may determine a_4 (up to sign, which we can deal with by changing signs of x_1, a_2, a_3) and then, in turn, $b_1, b_2 b_3$, $a_2 b_3 + a_3 b_2$ (recall $a_4 \neq 0$!), a_1 (recall $b_2 b_3 \neq 0$!) and $a_2 b_3 - a_3 b_2$. As $b_2 = b_3$, $b_2 b_3$ determines it up to sign (which we can deal with by taking $\rho = -1$); then we know $a_2 + a_3$, $a_2 - a_3$ and thus have a unique net. Recall this is so for all Σ meeting $\lambda = 0$ in two distinct points: Σ may not be a repeated line, or a line-pair with vertex on $\lambda = 0$.

If $a_1^2 + 2a_2 a_3 = 0$, Σ is a line-pair with vertex
$$(a_4, -a_1 a_4, a_1 b_1 + a_2 b_3 + a_3 b_2),$$

as it reduces to $\{a_4\nu+b_1\mu-(a_2b_3+a_3b_2)\lambda\}^2+2b_2b_3\{\mu+a_1\lambda\}^2$. If Δ has four distinct nodes, the further node of Δ corresponds to the quadric

$$a_1a_4S_2+(a_1b_1+a_2b_3+a_3b_2)(a_1S_0-S_1).$$

The cofactor of the first term is $a_1^3a_4^2b_2b_3(a_1b_1+a_2b_3+a_3b_2)^2\neq 0$ since $b_2b_3\neq 0$ and the vertex of Σ does not lie on $\lambda\mu\nu=0$. Thus we have another cone.

1.2.3. *Case of a plane-pair and two cones*

Take the plane-pair at N, and the pencil $\nu=0$ in standard form for type [2, 2]: $S_0=x_2^2+2x_3x_1$, $S_1=x_3^2+2x_2x_4$. Then (lemma 1.1) S_2 contains a plane through the vertex of each; these cannot be the same plane else the line $x_2=x_3=0$ would be common to the net. Thus we take

$$S_2=2(a_1x_1+a_2x_2+a_3x_3)(b_2x_2+b_3x_3+b_4x_4).$$

Another straightforward but tedious calculation yields

$$\Sigma=\begin{bmatrix}2a_1^2b_2b_4+a_1^2b_3^2 & a_1a_2b_3b_4-2a_1b_2a_3b_4-\tfrac{1}{2}a_1^2b_4^2 & a_1b_3\\ a_1a_2b_3b_4-2a_1b_2a_3b_4-\tfrac{1}{2}a_1^2b_4^2 & a_2^2b_4^2+2a_1a_3b_4^2 & a_2b_4\\ a_1b_3 & a_2b_4 & 1\end{bmatrix},$$

$$\det\Sigma=-a_1^2b_4^2\{2b_2a_3-\tfrac{1}{2}a_1b_4\}^2.$$

Note $a_1\neq 0$, $b_4\neq 0$ and $a_2^2+2a_1a_3\neq 0$, $b_3^2+2b_2b_4\neq 0$. So Σ is singular if and only if $a_1b_4=4a_3b_2$, and the respective conditions for cusps at L, M and N reduce to $a_3=0$, $b_2=0$ and $(\tfrac{1}{2}a_1b_4-2a_3b_2)^2=(a_1b_3+2a_2b_2)(a_2b_4+2a_3b_3)$. If all three are cusps, $a_3=b_2=0$ and $a_1b_4=4a_2b_3$. Now Σ determines a_1b_3, a_2b_4, $a_1a_3b_4^2$, $a_1^2b_2b_4$ and $a_1^2b_3^2+4a_1b_2a_3b_4$: say these have values α, β, γ, δ, ϵ. We can suppose $a_1=1$ (replace a_i by $a_ia_1^{-1}$, b_i by b_ib_1): then $b_3=\alpha$, $a_2=\beta x^{-1}$, $a_3=\gamma x^{-2}$, $b_2=\delta x^{-1}$ (where $x=b_4$) and $\epsilon=x^2+4\gamma\delta x^{-2}$; thus $x^4-\epsilon x^2+4\gamma\delta=0$. Each value of x yields a unique net; moreover, $\pm x$ yield the same (change the signs of x_2 and x_4). These two nets are in general different, but coincide if $\epsilon^2=16\gamma\delta$, i.e. if Σ is singular. Moreover, only non-zero roots are acceptable: both roots are non-zero for $\gamma\delta\neq 0$ – i.e. L, M not cusps – otherwise one is zero. We thus have 2 nets if L, M not cusps, Σ non-singular; 1 net otherwise, with Σ non-singular, or for Σ a line pair with vertex not on $\lambda\mu=0$; otherwise no nets.

When $a_1b_4=4a_3b_2$, Σ becomes

$$(\nu+a_2b_4\mu+a_1b_3\lambda)^2+2b_2b_4(a_1\lambda-2a_3\mu)^2=0,$$

with vertex at $(2a_3, a_1, -a_1(2a_3b_3+a_2b_4))$. When these coefficients are all non-zero, Δ has a fourth node, corresponding to $2a_3S_2-(a_2b_4+2a_3b_3)(a_1S_0+2a_3S_1)$, or equivalently to

$$(\lambda,\mu,\nu)=(a_1b_3+2a_2b_2, a_2b_4+2a_3b_3, -1).$$

This yields the plane-pair

$$(\tfrac{1}{2}a_1x_2-a_2x_3+2a_3x_4)(2b_2x_1-b_3x_2+\tfrac{1}{2}b_4x_3)=0.$$

1.2.4. *Case of three cones*

This case is distinctly more complicated than the preceding. We again take $S_0=x_2^2+2x_1x_3$, $S_1=x_3^2+2x_2x_4$. The vertex of S_2 lies on S_0 and S_1, but not on the line $x_2=x_3=0$ (else this would be common to the net), hence is of the form $(\alpha^3, -2\alpha^2\beta, -2\alpha\beta^2, \beta^3)$: we normalize coordinates by taking it as $(1, -2, -2, 1)$. The equation of S_2 can thus be expressed as a quadratic in $2x_1+x_2$ $2x_4+x_3$ and x_2-x_3: say

$$0 = a(2x_1+x_2)^2+b(2x_4+x_3)^2+c(x_2-x_3)^2$$
$$+2f(2x_4+x_3)(x_2-x_3)+2g(2x_1+x_2)(x_3-x_2)+2h(2x_1+x_2)(2x_4+x_3).$$

Since S_2 passes through X_1, X_4 we have $a = b = 0$; since it does not contain the line X_1X_4, $h \neq 0$: indeed, since it is a cone we have $ch+2fg \neq 0$ also.

The evaluation of the determinant for Δ is less tedious if we recall that we can ignore terms divisible by ν^3, which are bound to cancel. After reduction, we obtain

$$\Sigma = \begin{bmatrix} 4(g-h)^2 & 4H & 2(g+h) \\ 4H & 4(f-h)^2 & 2(f+h) \\ 2(g+h) & 2(f+h) & 1 \end{bmatrix},$$

where $H = -h^2+h(f+g)+fg+ch$. Since $h \neq 0$ and $ch+2fg \neq 0$, the respective conditions for cusps at L, M and N reduce to

$$f = 0, \quad g = 0 \quad \text{and} \quad c+2(f+g-h) = 0.$$

$$\det \Sigma = 16 \det \begin{bmatrix} -4gh & ch-2h^2 \\ ch-2h^2 & -4fh \end{bmatrix} = 16h^2\{16fg-(c-2h)^2\}.$$

We observe that if L, M, N are all cusps, $f = g = 0$ and $c = 2h$. But then $\det \Sigma = 0$. Thus this case does not arise.

We now consider enumeration. Since we have fixed coordinates more rigidly than usual, we should allow $S_0 = d(x_2^2+2x_1x_3)$, $S_1 = e(x_3^2+2x_2x_4)$. Then if

$$\Sigma = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \delta & \epsilon \\ \gamma & \epsilon & \zeta \end{bmatrix},$$

we have $\zeta = d^2e^2$, $\epsilon = 2d^2e(f+h)$, $\gamma = 2de^2(g+h)$, $\delta = 4d^2(f-h)^2$, $\alpha = 4e^2(g-h)^2$ and $\beta = 4deH$. Choosing signs of square roots of α, δ and ζ then yields values for de, $d(f-h)$ and $e(g-h)$ hence also (as $de \neq 0$) of $d(f+h)$ and $e(g+h)$, so of df, dh, eg and eh. Once de, dh and eh are known (and all non-zero) a further square root determines deh, hence d, e, h and then f and g; finally β yields a unique value for c. This appears to yield 16 cases, but as c, d, e, f, g and h may be multiplied by a common fourth root of unity without affecting Σ, this number should be reduced to 4.

However, if M is a cusp ($\alpha\zeta = \gamma^2$), one system of choices for square roots will yield $eh = 0$, so the number of cases is reduced from 16 to 8, hence finally from 4 to 2; similarly if L is a cusp, whereas if both occur we end with only one net. The same occurs if N is a cusp (we noted above that one system of choices of signs leads to a contradiction); we have already seen that all three cannot be cusps.

These conclusions hold also for the singular case (where of course we may only have one cusp, else Σ is a repeated line, leading to the same contradiction as for three cusps).

For future reference, we now summarize the enumerations of this section.

THEOREM 1.3. *For Δ with double points at* L, M, N, *but no linear components, the enumeration of (stable) net is as follows*:

S_0, S_1, S_2 plane-pairs: one net for each Δ.

S_0 a cone S_1, S_2 plane-pairs: no nets if L *is a cusp (or worse), otherwise one net.*

S_0, S_1 cones S_2 a plane-pair: two nets if Δ irreducible and L, M *simple nodes, one net if Δ irreducible otherwise, or two conics touching (if at all) at* N; *otherwise none.*

S_0, S_1, S_2 cones: four nets if Δ has no cusps, two if one cusp, (including case of two conics touching), one if two cusps and none if three cusps.

Moreover, if Δ consists of two conics meeting in four points, the number of points corresponding to plane-pairs is even.

This final conclusion is the first of several parity conditions which will be of considerable interest in later sections.

1.3. Δ *with a higher double point: no singular pencil*

If Δ has a double point at N, with coincident tangents $\lambda = 0$, we can write

$$\Delta = \nu^2\lambda^2 + \nu f_3(\lambda, \mu) + f_4(\lambda, \mu),$$

where f_i is homogeneous of degree i. If the coefficient of $\nu\mu^3$ is non-zero, we just have a cusp. Now write

$$\tfrac{1}{2}f_3(\lambda, \mu) = c\lambda^3 + d\lambda^2\mu + e\lambda\mu^2;$$

then the substitution $\nu' = \nu + c\lambda + d\mu$ reduces c and d to zero. We thus take as our normal form

$$\Delta = (\nu\lambda + e\mu^2)^2 + F_4(\lambda, \mu).$$

We shall suppose that Δ has no linear factor, i.e. that the coefficient of μ^4 in F_4 is not $-e^2$, but not that Δ is irreducible. Somewhat surprisingly, it is relatively unimportant whether or not e vanishes; cases will be classified according to the root pattern of F, noting that λ plays a special rôle.

We can easily list the types of singular point that occur. I will use Arnold's (1972) notation for these. Then a singular point is of the indicated simple type if and only if the function can be reduced by local analytic coordinate change to

$$\mathrm{A}_n: x^2 + y^{n+1}$$

$$\mathrm{D}_n: x^2y + y^{n-1}$$

$$\mathrm{E}_6: x^3 + y^4 \quad \mathrm{E}_7: x^3 + xy^3 \quad \mathrm{E}_8: x^3 + y^5$$

and we will need also the unimodular type $\tilde{\mathrm{E}}_7: x^4 + \lambda x^2y^2 + y^4$. We shall be mainly concerned with the double points A_n: for example, A_1 is a simple node, A_2 a cusp, A_3 a tacnode, A_4 a rhamphoidal cusp and A_5 an oscnode.

A factor $\mu + a\lambda$ of multiplicity r in F_4 meets the conic $\nu\lambda + e\mu^2 = 0$ (or if $e = 0$, the line $\nu = 0$) in a singular point of type A_{r-1} whereas if the factor λ itself has multiplicity r (here $e \neq 0$ if $r > 0$), the singular point N is of type A_{3+r} $(0 \leqslant r \leqslant 4)$. If the root pattern of F is of type 22 or 4, F is a perfect square and Δ splits into two conics. We even allow $F = 0$ (provided $e \neq 0$), for a repeated conic, but only consider stable nets.

The line $\lambda = 0$ has fourfold intersection with Δ at N. It thus determines a pencil of type [4] [(3,1)] or [(2,2)]. We treat these cases in turn.

1.3.1. If S_2 is a cone, take its equation as $x_2^2 + 2x_1x_3 = 0$. Evaluating det $(\lambda\mathrm{A}_0 + \mu\mathrm{A}_1 + \nu\mathrm{A}_2)$, the coefficient of ν^3 is $a_{44}\lambda + b_{44}\mu$, so $a_{44} = b_{44} = 0$. The coefficient of ν^2 is now $(a_{24}\lambda + b_{24}\mu)^2 + 2(a_{14}\lambda + b_{14}\mu)(a_{34}\lambda + b_{34}\mu)$, which must equal λ^2. Thus the polar plane $\Sigma b_{i4}x_i$ of the vertex X_4 of S_2 with respect to S_1 touches S_2, and $\Sigma a_{i4}x_i$ passes through its line of contact. We change coordinates to take these planes as x_3 and x_2 respectively; this fixes x_1, x_2 and x_3 (up to scalars). Now set $x_4' = x_4 + \alpha x_1 + \beta x_2 + \gamma x_3$ and choose α, β, γ (uniquely) so that $a_{12} = 0$, $b_{23} = 0$ and $a_{23} = 0$. This fixes coordinates, and now

$$\Delta = \det \begin{bmatrix} a_{11}\lambda + b_{11}\mu & b_{12}\mu & a_{13}\lambda + b_{13}\mu + \nu & 0 \\ b_{12}\mu & a_{22}\lambda + b_{22}\mu + \nu & 0 & \lambda \\ a_{13}\lambda + b_{13}\mu + \nu & 0 & a_{33}\lambda + b_{33}\mu & \mu \\ 0 & \lambda & \mu & 0 \end{bmatrix}$$

$$= -\lambda^2(a_{11}\lambda + b_{11}\mu)(a_{33}\lambda + b_{33}\mu) - \mu^2(a_{11}\lambda + b_{11}\mu)(a_{22}\lambda + b_{22}\mu + \nu)$$
$$+ \{a_{13}\lambda^2 + b_{13}\lambda\mu - b_{12}\mu^2 + \lambda\nu\}^2.$$

For the terms involving ν to be as desired, $a_{13} = b_{13} = 0$, $b_{11} = 0$ and $2e = -a_{11} - 2b_{12}$. Then

$$-F = a_{11}\{a_{33}\lambda^4 + b_{33}\lambda^3\mu + a_{22}\lambda^2\mu^2 + b_{22}\lambda\mu^3 + (b_{12} + \tfrac{1}{4}a_{11})\mu^4\}.$$

Note that if $a_{11} = 0$, the line $x_2 = x_3 = 0$ is common to all quadrics of the net, which is not stable.

We solve as follows. The coefficient of μ^4 in Δ (not in F) determines b_{12} up to sign: as it is non-zero, there are two cases. Next, $2e = -a_{11} - 2b_{12}$ determines a_{11}: at least one value for which is non-zero. The remaining coefficients of F then determine the remaining parameters uniquely. We obtain two nets except when the coefficient of μ^4 in Δ is e^2, i.e. when λ divides F, and then there is only one net.

These nets contain no plane-pairs; for the 3×3 minor which is cofactor to a_{33} equals $-a_{11}\lambda^3$, and we have seen $a_{11} \neq 0$. On $\lambda = 0$ there are no plane-pairs by hypothesis. The net is stable even when Δ is a repeated conic. In this case, it is the net of quadrics through a twisted cubic.

1.3.2. We turn to the case when S_2 is a plane-pair, and take it as $2xy$. The coefficient of ν^2 in Δ is then (minus) the discriminant of the pencil cut by $\lambda S_0 + \mu S_1$ on $x = y = 0$. We can thus choose z and t so that this pencil has matrix $\begin{bmatrix} c\mu & \lambda \\ \lambda & 0 \end{bmatrix}$. To normalize further, set $z' = z + \alpha x + \beta y$, $t' = t + \gamma x + \delta y$ and choose α, β, γ and δ to make $a_{14} = a_{24} = a_{13} = a_{23} = 0$ (where, as above, S_1 has matrix a_{ij} and S_2 matrix b_{ij}).

Then

$$\Delta = \det \begin{bmatrix} a_{11}\lambda + b_{11}\mu & a_{12}\lambda + b_{12}\mu + \nu & b_{13}\mu & b_{14}\mu \\ a_{12}\lambda + b_{12}\mu + \nu & a_{22}\lambda + b_{22}\mu & b_{23}\mu & b_{24}\mu \\ b_{13}\mu & b_{23}\mu & c\mu & \lambda \\ b_{14}\mu & b_{24}\mu & \lambda & 0 \end{bmatrix}.$$

Half the coefficient of ν must equal $e\lambda\mu^2$. Thus

$$-e\lambda\mu^2 = \det \begin{bmatrix} a_{12}\lambda + b_{12}\mu & b_{23}\mu & b_{24}\mu \\ b_{13}\mu & c\mu & \lambda \\ b_{14}\mu & \lambda & 0 \end{bmatrix}.$$

Equating the coefficients of $\lambda^3, \lambda^2\mu$ shows $a_{12} = b_{12} = 0$; and from the coefficient of μ^3, $b_{14}b_{24}c = 0$. For a pencil of type [(3, 1)] $c \neq 0$. The rôles of b_{14} and b_{24} are symmetrical (interchange x_1 and x_2) and we suppose $b_{24} = 0$. Then $e = -b_{23}b_{14}$, and

$$\Delta = (\lambda\nu - b_{14}b_{23}\mu^2)^2 - (a_{22}\lambda + b_{22}\mu)\{a_{11}\lambda^3 + b_{11}\lambda^2\mu - 2b_{13}b_{14}\lambda\mu^2 + b_{14}^2c\mu^3\}.$$

Thus one of the roots of F is preferred. Since we cannot have $a_{22} = b_{22} = 0$ (again an unstable case: $x_1 = x_3 = 0$ is here a common line), and are still free to normalize coordinates by $x_1 \to \alpha x_1$,

$x_2 \to \alpha^{-1}x_2$, $x_3 \to \beta x_3$, $x_4 \to \beta^{-1}x_4$ we can use α to adjust the multiplicative constant in the factor $a_{22}\lambda + b_{22}\mu$ uniquely. The coefficients in Δ now determine $a_{11}, b_{11}, b_{14}b_{23}, b_{13}b_{14}$ and $b_{14}^2 c$. We can use β to adjust $b_{14} = 1$ and then solve uniquely for b_{13}, b_{23} and c. There cannot be a solution with $b_{14} = 0$ here, since this would imply Δ divisible by λ. The requirement $c \neq 0$ implies that none of the unselected factors of F is λ. Subject to this, we have one net for each choice of factor.

The cofactor of the leading term equals $-\lambda^2(a_{22}\lambda + b_{22}\mu)$. For a plane pair (other than S_2, on $\lambda = 0$) this must vanish, so $a_{22}\lambda + b_{22}\mu = 0 = \lambda\nu + e\mu^2$. Conversely, if $a_{22}\lambda + b_{22}\mu$ is indeed a repeated factor of F, then by lemma 1.2, the rank of the corresponding quadric is indeed 2.

1.3.3. For the final case – a pencil of type [(2, 2)] – we have $c = 0$ above, and may not assume that b_{24} (or b_{14}) vanishes. The other normalizations still hold, and $-e = b_{14}b_{23} + b_{13}b_{24}$. The terms not involving ν form

$$\det \begin{bmatrix} a_{11}\lambda + b_{11}\mu & 0 & b_{13}\mu & b_{14}\mu \\ 0 & a_{22}\lambda + b_{22}\mu & b_{23}\mu & b_{24}\mu \\ b_{13}\mu & b_{23}\mu & 0 & \lambda \\ b_{14}\mu & b_{24}\mu & \lambda & 0 \end{bmatrix}$$

so

$$\Delta = \{\lambda\nu - (b_{14}b_{23} + b_{13}b_{24})\,\mu^2\}^2 - \{a_{11}\lambda^2 + b_{11}\lambda\mu - 2b_{13}b_{14}\mu^2\}\{a_{22}\lambda^2 + b_{22}\lambda\mu - 2b_{23}b_{24}\mu^2\}.$$

This time we have a partition of the factors of F into two sets of two. The order is insignificant: we can interchange x and y. Neither factor may vanish identically. For if, for example, $a_{11} = b_{11} = b_{13} = 0$, then $x_2 = x_4 = 0$ is a base line of the system, which is unstable.

Now suppose given Δ (not a repeated conic) and a partition as above. We may use $x_1' = \alpha x_1$, $x_2' = \alpha^{-1}x_2$, $x_3' = \beta x_3$, $x_4' = \beta^{-1}x_4$ again for normalisation. Using α, we can adjust the factors by reciprocal scalars, and so suppose known

$$a_{11}, b_{11}, a_{22}, b_{22}, b_{13}b_{14}, b_{23}b_{24} \quad \text{and} \quad b_{14}b_{23} + b_{13}b_{24}.$$

We thus know $(b_{14}b_{23} - b_{13}b_{24})^2 = (b_{14}b_{23} + b_{13}b_{24})^2 - 4b_{13}b_{14}b_{23}b_{24}$, hence $b_{14}b_{23} - b_{13}b_{24}$ up to sign, and hence $b_{14}b_{23}$ and $b_{13}b_{24}$ up to a possible interchange. As this interchange can be effected by interchanging x_3 and x_4, we may suppose all four products known.

If any of the four – say $b_{13}b_{14} = t$ – is non-zero, we can use β to normalize $b_{13} = 1$, $b_{14} = t$ and then uniquely determine $b_{24} = b_{13}b_{24}$ and $b_{23} = t^{-1}b_{13}b_{23}$. However, if $b_{13}b_{14} = b_{23}b_{24} = b_{14}b_{23} = b_{13}b_{24} = 0$, then λ^2 is a factor of Δ: an excluded case. Thus in all cases we obtain a unique net.

To seek further plane-pairs in the net, equate to zero the cofactors of the (1, 1) and (2, 2) terms; obtaining

$$0 = -\lambda(a_{22}\lambda^2 + b_{22}\lambda\mu - 2b_{23}b_{24}\mu^2) = -\lambda(a_{11}\lambda^2 + b_{11}\lambda\mu - 2b_{13}b_{14}\mu^2).$$

Since $\lambda \neq 0$, $\lambda : \mu$ must belong to *both* sets of factors of F. Conversely, by lemma 1.2 the intersection with $\lambda\nu + e\mu^2 = 0$ of such a line does indeed correspond to a plane-pair.

The enumerations of this section may be summarized as follows:

THEOREM 1.4. *For Δ with a higher double point and no linear factor, nets are enumerated as follows:*

If the pencil $\lambda = 0$ has type [4] *there are two nets for each Δ, except when λ divides F_4 when there is one net.*

For type [(3, 1)], *there is one net for each selection of a linear factor of F_4, provided λ is not an unselected factor.*

For type [(2, 2)], *there is one net for each factorization of F_4 into two quadratic factors.*

1.4. Δ with triple point; no singular pencil

There are three essentially distinct cases, determined by coincidences of tangents at the triple point. We normalize the triple point as N, so that $\Delta = \nu f_3(\lambda, \mu) + f_4(\lambda, \mu)$ where we can take f_3 as $\lambda\mu(\lambda+\mu)$, $\lambda^2\mu$ or λ^3 respectively in the three cases mentioned. The point is correspondingly a singularity of type D_4, D_5 or E_6. Δ is irreducible if and only if f_3 and f_4 have no common factor. There can be no further multiple point, else the line joining the two would have five intersections with Δ.

First suppose S_2 a cone, with vertex X_4. If u_0, u_1 are the polar planes of X_4 with respect to S_0, S_1 then (as in the case above) since the coefficient of ν^2 vanishes, each $\lambda u_0 + \mu u_1$ is a tangent plane to S_2 through the vertex. This implies that some such expression vanishes identically. But then X_4 is also a vertex of $\lambda S_0 + \mu S_1$, and we have a subpencil with common vertex. Hence this case cannot arise.

Next take S_2 as the plane-pair $2x_1 x_2$. Again as previously, since the coefficient of ν^2 vanishes the trace of $\lambda S_0 + \mu S_1$ on $x_1 = x_2 = 0$ must be a degenerate point-pair for each λ, μ. This cannot vanish, else the line would be common to all quadrics of the net, which is then unstable. Hence there is a fixed point – say X_4 – on the line. We now have

$$\Delta = \det \begin{bmatrix} a_{11}\lambda + b_{11}\mu & a_{12}\lambda + b_{12}\mu + \nu & a_{13}\lambda + b_{13}\mu & a_{14}\lambda + b_{14}\mu \\ a_{12}\lambda + b_{12}\mu + \nu & a_{22}\lambda + b_{22}\mu & a_{23}\lambda + b_{23}\mu & a_{24}\lambda + b_{24}\mu \\ a_{13}\lambda + b_{13}\mu & a_{23}\lambda + b_{23}\mu & a_{33}\lambda + b_{33}\mu & 0 \\ a_{14}\lambda + b_{14}\mu & a_{24}\lambda + b_{24}\mu & 0 & 0 \end{bmatrix}.$$

The coefficient of ν is $2(a_{14}\lambda + b_{14}\mu)\ (a_{24}\lambda + b_{24}\mu)\ (a_{33}\lambda + b_{33}\mu)$, so these terms give the tangents to Δ at N. Observe that if $a_{14}\lambda + b_{14}\mu$, $a_{24}\lambda + b_{24}\mu$ represent the same tangent, they are proportional, so Δ is divisible by $(a_{14}\lambda + b_{14}\mu)^2$ (in fact, the net is unstable). So these are distinct tangents: normalize them as λ and μ respectively, and suppose (interchanging λ and μ, x_1 and x_2 if necessary) $a_{33} \neq 0$. Normalize coordinates further by

$$x_3' = x_3 + \alpha x_1 + \beta x_2, \quad x_4' = x_4 + \gamma x_1 + \delta x_2 + \epsilon x_3,$$

where we choose α, β, γ, δ and ϵ to make

$$a_{13} = 0, a_{23} = 0, b_{12} = 0, a_{12} = 0, b_{23} = 0.$$

Then

$$\Delta = \det \begin{bmatrix} a_{11}\lambda + b_{11}\mu & \nu & b_{13}\mu & \lambda \\ \nu & a_{22}\lambda + b_{22}\mu & 0 & \mu \\ b_{13}\mu & 0 & a_{33}\lambda + b_{33}\mu & 0 \\ \lambda & \mu & 0 & 0 \end{bmatrix}$$

$$= b_{13}^2\mu^4 + (a_{33}\lambda + b_{33}\mu)\{2\lambda\mu\nu - \lambda^2(a_{22}\lambda + b_{22}\mu) - \mu^2(a_{11}\lambda + b_{11}\mu)\}.$$

Then given $f_4(\lambda, \mu)$, as we know $a_{33}\lambda + b_{33}\mu$ and $a_{33} \neq 0$, the coefficients $a_{22}, b_{22}, a_{11}, b_{11}$ are uniquely determined, and hence also b_{13}^2. The choice of sign of b_{13} is cancelled by changing the sign of the coordinate x_3, so we have a unique net. More precisely, given Δ choose one of the tangents at N to play the role of $a_{33}\lambda + b_{33}\mu$. The other two must be distinct: call them λ and μ. Their roles are symmetrical (we can interchange x_1 and x_2). The argument now yields a unique net.

Lastly suppose $S_2 = x_4^2$ a repeated plane. The coefficient of ν is then the discriminant of the pencil of conics cut on S_2 by S_0 and S_1. For the case of distinct tangents, the pencil has type [1, 1, 1] and normalizing coordinates in $x_4 = 0$, we have

$$\begin{bmatrix} \lambda & 0 & 0 & a_{14}\lambda+b_{14}\mu \\ 0 & \mu & 0 & a_{24}\lambda+b_{24}\mu \\ 0 & 0 & \lambda+\mu & a_{34}\lambda+b_{34}\mu \\ a_{14}\lambda+b_{14}\mu & a_{24}\lambda+b_{24}\mu & a_{34}\lambda+b_{34}\mu & a_{44}\lambda+b_{44}\mu+\nu \end{bmatrix}.$$

Set $x_1' = x_1+\alpha x_4$, $x_2' = x_2+\gamma x_4$ and choose α, β, γ so that $0 = a_{14} = b_{24}$, $b_{34} = -a_{34}$. Then

$$\begin{aligned} \Delta &= \lambda\mu(\lambda+\mu)\,(a_{44}\lambda+b_{44}\mu+\nu) - b_{14}^2\mu^3(\lambda+\mu) \\ &\quad - a_{24}^2\lambda^3(\lambda+\mu) - a_{34}^2\lambda\mu(\lambda-\mu)^2 \\ &= \nu\lambda\mu(\lambda+\mu) - a_{24}^2\lambda^4 + \lambda^3\mu(a_{44}-a_{24}^2-a_{34}^2) \\ &\quad + \lambda^2\mu^2(a_{44}+b_{44}+2a_{34}^2) + \lambda\mu^3(b_{44}-b_{14}^2-a_{34}^2) - b_{14}^2\mu^4, \end{aligned}$$

which determines a_{44}, b_{44}, b_{14}^2, a_{24}^2 and a_{34}^2 uniquely. As usual, we can change signs of coordinates, so we have a unique net.

If there is a repeated tangent $(\lambda^2\mu)$, and we try a pencil of type [(1,1) 1], we find λ divides Δ. Hence the pencil must have type [(2,1)]:

$$\begin{bmatrix} 0 & \lambda & 0 & a_{14}\lambda+b_{14}\mu \\ \lambda & \mu & 0 & a_{24}\lambda+b_{24}\mu \\ 0 & 0 & \mu & a_{34}\lambda+b_{34}\mu \\ a_{14}\lambda+b_{14}\mu & a_{24}\lambda+b_{24}\mu & a_{34}\lambda+b_{34}\mu & a_{44}\lambda+b_{44}\mu+\nu \end{bmatrix},$$

where we can normalize $a_{14} = a_{24} = b_{34} = 0$. Thus

$$\begin{aligned} \Delta &= -\lambda^2\mu(a_{44}\lambda+b_{44}\mu+\nu) + \lambda^2(a_{34}\lambda)^2 + 2\lambda\mu(b_{14}\mu)\,(b_{24}\mu) - \mu^2(b_{14}\mu)^2 \\ &= -\lambda^2\mu\nu + a_{34}^2\lambda^4 - a_{44}\lambda^3\mu - b_{44}\lambda^2\mu^2 + 2b_{14}b_{24}\lambda\mu^3 - b_{14}^2\mu^4. \end{aligned}$$

Since the coefficient of μ^4 in Δ cannot vanish, we deduce again that we have a unique net.

Finally if all three tangents coincide, we have (as before) to exclude the pencil of type [(2,1)] (and the degenerate case [(1,1,1)]) so must have one of type [3],

$$\begin{bmatrix} 0 & 0 & \lambda & a_{14}\lambda+b_{14}\mu \\ 0 & \lambda & \mu & a_{24}\lambda+b_{24}\mu \\ \lambda & \mu & 0 & a_{34}\lambda+b_{34}\mu \\ a_{14}\lambda+b_{14}\mu & a_{24}\lambda+b_{24}\mu & a_{34}\lambda+b_{34}\mu & a_{44}\lambda+b_{44}\mu+\nu \end{bmatrix},$$

where we may normalize $a_{14} = a_{24} = a_{34} = 0$. Thus

$$\begin{aligned} \Delta &= -\lambda^3(a_{44}\lambda+b_{44}\mu+\nu) + 2\lambda^2(b_{14}\mu)(b_{34}\mu) + \lambda^2(b_{24}\mu)^2 - 2\lambda\mu(b_{14}\mu)(b_{24}\mu) + \mu^2(b_{14}\mu)^2 \\ &= -\lambda^3\nu - a_{44}\lambda^3 - b_{44}\lambda^3\mu + (b_{24}^2+2b_{14}b_{34})\,\lambda^2\mu^2 - 2b_{14}b_{24}\lambda\mu^3 + b_{14}^2\mu^4 \end{aligned}$$

and we may conclude, as before, that for each irreducible Δ we have a unique net.

Theorem 1.5. *If Δ is irreducible, with triple point* N, *there are no nets with S_2 a cone; for S_2 a plane-pair there is one net for each choice of tangent at* N *such that the other two tangents are distinct: for S_2 a repeated plane, we have a unique net.*

1.5. *Case of singular subpencil*

1.5.1. This splits into two main subcases. First suppose the singular subpencil has a common vertex. We may take the subpencil as $\lambda = 0$ and the vertex as X_1. If X_1 lies on S_0, either it is a vertex and the net is not semistable, or we can take the tangent plane there as $x_4 = 0$ and have one standard case for unstable nets. Otherwise, let $x_1 = 0$ be the polar plane of X_1 with respect to S_0, so we can write

$$S_0 = x_1^2 + S_0'$$

where S_0', like S_1 and S_2, does not involve x_1. We can thus classify this as a net (possibly degenerating to a pencil) of conics, but noting that the subpencil $\lambda = 0$ plays a special role; if the discriminant of the net of conics is Δ_1, then $\Delta = \lambda\Delta_1$. Clearly this net, plus the specification of a subpencil, determines up to isomorphism our original net.

If the plane net has $\lambda = 0$ as a singular subpencil, λ^2 divides Δ_1. If the subpencil has a common vertex $x_3 = x_4 = 0$, then any point of the line $x_3 = x_4 = 0$ is a common vertex of the quadrics S_1 and S_2, and some point of the line lies on S_0, so our net is not semistable.

We recall from 1.1 the features of the classification of nets of conics. There are four types with Δ_1 irreducible (A, B, B*, C); five types where Δ_1 is reducible but has no repeated factor (D, D*, E, E*, F*); four semistable types (F, G, G*, H) and two (I, I*) which are not semistable. First consider cases where Δ has no repeated factor. For each case $\lambda\Delta_1$ with Δ_1 irreducible, we find one net for each net of conics corresponding to Δ_1. However, when Δ_1 is reducible, the choice of λ may not be unique. Now the singular subpencils of nets of conics of types D, E, F* do have common vertices; those of D*, E* do not. Thus if Δ has irreducible factors of degrees 2, 1, 1 we may choose either or both of the lines to correspond to pencils with common vertex, provided that a singular line tangent to the conic does so. If Δ has four linear factors, either just one has a common sub-pencil, and we have a net of type E*, the others forming a triangle; or all four have common sub-pencils, and we have either a net of type E or a degenerate net with one conic given by $0 = 0$.

Now consider cases when Δ has a repeated factor, but the net is stable. If the net corresponding to Δ_1 is not stable, we may suppose all the conics of the net have tangent $x_2 = 0$ at X_4. But then the same holds for our net of quadrics, which is then also unstable. There remain only the cases when the net has type D* or E* and we 'add' as subpencil a pencil already singular. It may be verified directly that these nets are both stable.

There are thus just three types of stable net such that Δ contains a repeated component: the two just obtained, and the net of quadrics through a twisted cubic. The two new ones:

$$2\lambda x_1 x_3 + 2\mu x_2 x_3 + \nu(2x_1 x_2 + x_3^2 + x_4^2)$$

$$2\lambda x_1 x_3 + 2\mu x_2 x_3 + \nu(2x_1 x_2 + x_4^2),$$

both consist of quadrics with a common conic $x_3 = 0$, $2x_1 x_2 + x_4^2 = 0$, and it seems apposite to enquire about nets whose base locus has positive dimension. If this locus contains a plane the net is not semistable; if a line, it is not stable. The only remaining possibilities are a twisted cubic, yielding the unique net just mentioned, and a conic. Nets through the above conic are all contained in the linear system

$$x_3(2\alpha x_1 + 2\beta x_2 + \gamma x_3 + 2\delta x_4) + \epsilon(2x_1 x_2 + x_4^2),$$

with discriminant $\epsilon^2(2\alpha\beta + \delta^2 - \gamma\epsilon)$, so Δ always has a repeated factor. Observe that such a net contains a pencil formed by x_3 with a pencil of planes. If the axis of this meets the conic, this point

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has a common tangent plane for all quadrics of the system, which is then unstable. If the axis meets $x_3 = 0$ at a point not on the conic, we can choose the point as X_4 and then quickly reduce to one of the two above nets.

We tabulate the results from this subcase as

THEOREM 1.6. *Stable nets having a subpencil with common vertex are classified as follows: if $\Delta = \lambda\Delta_1$ with Δ_1 irreducible, we have one net for each net of conics with discriminant Δ_1. If Δ consists of a conic and two lines, or of four lines the net is unique when we know which lines correspond to pencils with common vertex. In the former case this includes at least one line, and any which touch the conic. In the latter case we either have just one line (the others forming a triangle) or all four.*

There are just three types of stable net corresponding to Δ with a repeated component; in each case, the base locus has dimension 1.

1.5.2. Now suppose there is a singular subpencil, but none such with a common vertex. As there is only one such type of pencil in this dimension, we can take

$$S_0 = x_1^2 + 2x_2 x_3 \quad S_1 = 2x_2 x_4 \quad S_2 = \sum_{i,j=1}^{4} a_{ij} x_i x_j.$$

We first compute

$$\Delta = \det \begin{bmatrix} \lambda + a_{11}\nu & a_{12}\nu & a_{13}\nu & a_{14}\nu \\ a_{12}\nu & a_{22}\nu & \lambda + a_{23} & \mu + a_{24}\nu \\ a_{13}\nu & \lambda + a_{23}\nu & a_{33}\nu & a_{34}\nu \\ a_{14}\nu & \mu + a_{24}\nu & a_{34}\nu & a_{44}\nu \end{bmatrix},$$

$$\begin{aligned}\Delta = \nu\{-a_{44}\lambda^3 + 2a_{34}\lambda^2\mu - a_{33}\lambda\mu^2\} + \nu^2\{(a_{14}^2 - a_{11}a_{44} + 2a_{24}a_{34} - 2a_{23}a_{44})\lambda^2 \\ + (2a_{11}a_{34} - 2a_{13}a_{14} + 2a_{23}a_{34} - 2a_{24}a_{33})\lambda\mu + (a_{13}^2 - a_{11}a_{33})\mu^2\} \\ + \nu^3\{(A_{11} - 2A_{23})\lambda + 2A_{24}\mu\} + (\det S_2)\nu^4.\end{aligned}$$

It is clear from this that the intersections of S_2 with the line $x_1 = x_2 = 0$ on $S_0 \cap S_1$ are important to the classification: it is also important to know if either lies on the other plane $x_4 = 0$ of S_1. Since the line may not lie on S_2, we have four cases, according as the line meets S_2 in two points or touches it, and whether X_3 lies on S_2.

For enumeration, we also need to observe that the normalization of S_0 and S_1 does not yet fix the coordinates x_i, even up to scalar multiples: we may still substitute

$$x_1' = x_1 + \rho x_2, \quad x_2' = x_2, \quad x_3' = -\rho x_1 - \tfrac{1}{2}\rho^2 x_2 + x_3, \quad x_4' = x_4.$$

We can work out the effect on the a_{ij}; in particular,

$$a_{13}' = a_{13} - 2\rho a_{33} \quad \text{and} \quad a_{14}' = a_{14} - \rho a_{34}, \quad a_{11}' = a_{11} - 2\rho a_{13} + \rho^2 a_{33}.$$

In the two cases when $a_{33} \neq 0$ (i.e. X_3 is not on S_2) we will normalize coordinates by $a_{13} = 0$.

We now suppose $\Delta = \nu\Delta_1$, where Δ_1 is a given cubic through M:

$$\Delta_1 = b_1\lambda^3 + b_2\lambda^2\mu + b_3\lambda\mu^2 + b_4\lambda^2\nu + b_5\lambda\mu\nu + b_6\mu^2\nu + b_7\lambda\nu^2 + b_8\mu\nu^2 + b_9\nu^3,$$

and seek by equating coefficients to find how many nets of the above type have discriminant Δ.

First suppose $b_3 \neq 0$. Then $a_{33} = -b_3 \neq 0$ so, as above, we can normalize coordinates x_1, x_2, x_3, x_4 by $a_{13} = 0$. The only possible change remaining is $x_1' = -x_1$. Moreover, M is a simple point on Δ_1; the tangent there is not $\nu = 0$, so may be chosen as $\lambda = 0$; thus $b_6 = 0$.

We seek to solve for the a_{ij} the equations

$$b_1 = -a_{44},\quad b_2 = 2a_{34},\quad b_3 = -a_{33},\quad a_{13} = 0 \text{ (normalize)},$$
$$b_4 = a_{14}^2 - a_{11}a_{44} + 2(a_{24}a_{34} - a_{23}a_{44})$$
$$b_5 = 2(a_{11}a_{34} - a_{13}a_{14}) + 2(a_{23}a_{34} - a_{24}a_{33})$$
$$b_6 = a_{13}^2 - a_{11}a_{33}$$
$$b_7 = A_{11} - 2A_{23},\quad b_8 = 2A_{24}\quad \text{and}\quad b_9 = \det S_2.$$

Since a_{13} and b_6 are normalized to 0, this equation yields $a_{11} = 0$. Now suppose $a_{14}^2 = z$ known; then the equations for b_4 and b_5 may be solved for a_{23} and a_{24} in terms of z and the b_j, provided the discriminant $\delta = a_{33}a_{44} - a_{34}^2 = b_1 b_3 - \frac{1}{4}b_2^2$ does not vanish. Next, $A_{24} = a_{12}a_{14}a_{33}$ (since $a_{11} = a_{13} = 0$), so we obtain $a_{12}a_{14} = -b_8/2b_3$, and the equation for b_7 will determine a_{22} in terms of the remaining a_{ij} (since it has coefficient δ) provided $\delta \neq 0$. By eliminating all of these, the final equation yields an equation for z: if $z \neq 0$ we obtain a_{14}, and hence a_{12}, up to (simultaneous) change of sign, which corresponds to changing the sign of x_1. Thus for $a_{33} \neq 0$, $\delta \neq 0$ any value of $z \neq 0$ yields a unique net.

We now perform this elimination. Eliminate a_{22} by

$$\delta b_9 - A_{22}b_7 = \delta \det S_2 - A_{22}(A_{11} - 2A_{23})$$
$$= 2A_{22}A_{23} - A_{12}^2 \text{ (by a standard determinantal identity).}$$

But expansion of these minors yields

$$A_{12} = \delta a_{12} + \tfrac{1}{2}b_5 a_{14},\quad A_{22} = b_3 z,\quad A_{23} = -b_2 b_8/(4b_3) - za_{23}$$

so
$$\delta b_9 - b_3 b_7 z + \tfrac{1}{2}b_2 b_8 z + 2b_3 a_{23} z^2 + (\delta a_{12} + \tfrac{1}{2}b_5 a_{14})^2 = 0. \qquad (*)$$

Multiply by $b_3^2\delta z$, and recall $a_{12}a_{14} = -b_8/2b_3$, $a_{14}^2 = z$ and $2\delta a_{23} = b_3 b_4 - \frac{1}{2}b_2 b_5 - b_3 z$; then we have, on collecting terms,

$$(-b_3 z)^4 + (b_3 z)^3(b_3 b_4 - \tfrac{1}{2}b_2 b_5) + (b_3 z)^2 \delta(\tfrac{1}{4}b_5^2 + \tfrac{1}{2}b_2 b_8 - b_3 b_7) + (b_3 z)\,\delta^2(b_3 b_9 - \tfrac{1}{2}b_5 b_8) + \delta^3(\tfrac{1}{4}b_8^2) = 0.$$

Comparing this with the condition that $\lambda = w\nu$ touch Δ_1, namely $(b_2 w^2 + b_5 w + b_8)^2 = 4b_3 w(b_1 w^3 + b_4 w^2 + b_7 w + b_9)$, we see that it is equivalent to saying that $\delta\lambda + b_3 z\nu$ is tangent to Δ_1.

We turn to consideration of special cases. Provided b_3 and δ are non-zero, trouble can only arise if $z = 0$; and for this to be a root, $b_8 = 0$. Here if $b_9 \neq 0$, 0 is a simple root; now equation (*) determines $a_{12}^2 = -b_9/\delta$ and the others are determined as before. However, if $b_8 = b_9 = 0$, choosing $z = 0$ implies $a_{12} = a_{14} = 0$ and though we have a net, the pencil $\lambda = 0$ in it has a common vertex X_1. If there is a singular subpencil not through M, we take it as $\mu = 0$; thus $b_1 = b_4 = b_7 = b_9 = 0$; now $\delta \neq 0$ implies $b_2 \neq 0$ and hence $a_{34} \neq 0$. Thus S_2 cannot have the same vertex (X_4) as S_0. The pencil $\mu = 0$ is thus of the kind above, so just one quadric in it is a plane-pair; and this must be a double point of Δ, hence also of $\Delta_1 = \mu\Delta_2$, so is one of the (two) points $\mu = \Delta_2 = 0$. Now the 'tangents' from M to Δ_1, i.e. lines meeting it in two coincident points, must be lines from M to double points (there can be no 'honest' tangents). By lemma 1.2, the preferred tangent is the one through the plane-pair.

We must also consider the case when Δ_1 splits into three linear factors: λ, μ and a third $(b_2\lambda + b_3\mu + b_5\nu)$: our hypothesis still yields $b_2 \neq 0$, $b_3 \neq 0$. To avoid λ being singular we need $z \neq 0$; now z satisfies

$$(-b_3 z)^2 + (b_3 z)(-\tfrac{1}{2}b_2 b_5) + \tfrac{1}{4}b_5^2(\delta = -\tfrac{1}{4}b_2^2) = 0,$$

which reduces to $z = -b_2 b_5/4b_3$ (so we need $b_5 \neq 0$ also: no three of the lines concur), and obtain a unique net.

We can use the above formulae also to settle the case $b_3 \neq 0$, $\delta = 0$. We use the same normalizations. Also, the line $\nu = 0$ cuts Δ in two coincident points other than M. Choosing this point as L, we have $b_1 = b_2 = 0$. Hence $a_{34} = a_{44} = 0$ and our equations reduce to

$$a_{33} = -b_3, a_{13} = 0, \quad a_{11} = 0 \quad \text{and} \quad b_4 = a_{14}^2, \quad b_5 = -2a_{23}a_{33}, \quad a_{12}a_{14} = -b_8/2b_3,$$

$$b_7 = -a_{33}a_{24}^2 - 2(a_{14}^2 a_{23}) \quad \text{and} \quad b_9 = 2a_{12}a_{14}a_{24}a_{33} - a_{14}^2(a_{22}a_{33} - a_{23}^2).$$

If $b_4 \neq 0$, we can solve in turn b_4 for $a_{14} \neq 0$, b_8 for a_{12}, b_5 for a_{24}, b_7 for a_{23} and b_9 for a_{22}; thus there is a unique net. But if $b_4 = 0$ we deduce $a_{14} = 0$ and hence $b_8 = b_9 = 0$: there is usually no net, and even when there is, $S_2 - b_5/2b_3 S_1$ has the same vertex (X_1) as S_0, so the net is unstable.

Next, if Δ_1 factorizes we may suppose (as before) that λ or μ is a factor. But if μ is a factor, then $b_4 = 0$. If λ is a factor, $b_8 = b_9 = 0$. Since $a_{14} \neq 0$, $a_{12} = 0$. The line X_1X_3 meets S_2 twice at X_1, which is not a vertex of S_2; hence there is no vertex on this line. Thus this subpencil does not have a common vertex.

1.5.3. Next, suppose $b_3 = 0$ but $\delta \neq 0$. Here $b_2 \neq 0$ and we can choose the point L so that $b_1 = 0$. As $b_2 \neq 0$, $a_{34} \neq 0$. We cannot normalize the x coordinates by $a_{13} = 0$, but may suppose instead that $a_{14} = 0$.

Our system of equations now reduces to

$$a_{33} = a_{44} = a_{14} = 0, \quad b_2 = 2a_{34}(\neq 0),$$

$$b_4 = 2a_{24}a_{34}, \quad b_5 = 2a_{11}a_{34} + 2a_{23}a_{34}, \quad b_6 = a_{13}^2,$$

$$b_7 = A_{11} - 2A_{23} = 2a_{23}a_{24}a_{34} - a_{22}a_{34}^2 - 2\{-a_{11}a_{24}a_{34}\},$$

$$\tfrac{1}{2}b_8 = A_{24} = a_{11}a_{23}a_{34} + a_{13}^2 a_{24} - a_{12}a_{34}a_{13},$$

$$b_9 = \det S_2 = -a_{34}^2(a_{11}a_{22} - a_{12}^2) + 2a_{24}a_{34}(a_{11}a_{23} - a_{12}a_{13}) + a_{13}^2 a_{24}^2,$$

whence $$a_{34} = \tfrac{1}{2}b_2, a_{24} = b_4/b_2, a_{11} + a_{23} = b_5/b_2, a_{13} = \sqrt{b_6}$$

and $$b_7 = b_5 a_{24} - a_{22}a_{34}^2, \text{ so that } a_{22} = 4/b_2^3\,(b_5b_4 - b_2b_7)$$

$$b_9 - a_{24}b_8 = -a_{34}^2(a_{11}a_{22} - a_{12}^2) - a_{13}^2 a_{24}^2, \quad \text{so that}$$

$$a_{11}a_{22} - a_{12}^2 = 4b_2^{-4}\{b_2b_4b_8 - b_4^2b_6 - b_2^2b_9\};$$

finally $$a_{11}^2 - \frac{b_5}{b_2}a_{11} + \frac{b_8}{b_2} - \frac{2b_4b_6}{b_2^2} + \sqrt{b_6}\,a_{12} = 0 \quad (\text{from } b_8\text{-equation}).$$

If $b_6 \neq 0$, we can express a_{12} in terms of a_{11} from this last equation; substituting in the previous one then gives an equation for a_{11}, each solution of which corresponds to a net (note that the choice of sign of $\sqrt{b_6}$ corresponds again to that of x_1). Namely, we have

$$\frac{4b_6a_{11}}{b_2^3}(b_5b_4 - b_2b_7) - \left\{a_{11}^2 - \frac{b_5}{b_2}a_{11} + \frac{b_8b_2 - 2b_4b_6}{b_2^2}\right\}^2 = \frac{4\{b_2b_4b_8 - b_4^2b_6 - b_2^2b_9\}}{b_2^4}$$

which reduces to $$(b_2a_{11}^2 - b_5a_{11} + b_8)^2 = 4b_6(b_4a_{11}^2 - b_7a_{11} + b_9),$$

expressing the condition that $\lambda + a_{11}\nu = 0$ touches Δ_1. If $b_6 = 0$, this equation still holds; a_{12} must now be determined from the equation for $a_{11}a_{22} - a_{12}^2$, the ambiguity of sign corresponding (as usual) to the choice of sign of x_1.

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We must consider special cases. If M is a simple point of Δ_1, then $b_6 \neq 0$, so if Δ_1 contains a line through M it must be $\nu = 0$. But we have supposed $b_2 \neq 0$. For a line not through M, this must meet $\nu = 0$ at L, and we may take it as $\mu = 0$. This is the case $b_4 = b_7 = b_9 = 0$. Then $a_{12} = a_{22} = a_{14} = a_{24} = 0$, so $S_2 - a_{11} S_0$ has vertex X_1 in common with S_1, and the case must be excluded.

If M is a double point of Δ_1, then $b_6 = 0$. The tangent $\lambda + a_{11}\nu$ is now a tangent at the double point M. This is a component of Δ_1 if and only if we also have $b_4 a_{11}^2 - b_7 a_{11} + b_9 = 0$, or equivalently, $a_{11}(b_5 b_4 - b_2 b_7) = b_4 b_8 - b_2 b_9$. This coincides with the condition that $a_{12} = 0$ hence (as above) the pencil has a common vertex (certainly $S_2 - a_{11} S_0$ cannot have a vertex on $X_1 X_3$ other than at X_1). This always happens if Δ_1 is a product of three lines.

It remains to consider the case $b_3 = \delta = 0$, i.e. $b_2 = b_3 = 0$. Here we may suppose $b_1 \neq 0$, otherwise $a_{33} = a_{34} = a_{44} = 0$ and the line $x_1 = x_2 = 0$ is common to all quadrics: the case is not semistable. Arguing as before, we find that if $b_6 \neq 0$ we can solve for a_{13}, a_{14}, a_{23} and a_{24}, and then have two equations which yield the values of $a_{11} a_{23} - a_{12} a_{13}$ and $a_{12} a_{23} - a_{22} a_{13}$. Now as $a_{13} \neq 0$ we can normalize $a_{11} = 0$ and then determine a_{12}, a_{22}. We thus have a unique net. Observe that as $b_6 \neq 0$, M is a simple point of Δ_1; moreover $\nu = 0$ is an inflexional tangent. Thus Δ_1 is irreducible.

If, however, $b_6 = 0$ (so that M is a double point of Δ_1) $b_6 = a_{13}^2$ implies $a_{13} = 0$ and now $b_5 = -2a_{13} a_{14}$, $b_8 = 2a_{13}(a_{13} a_{24} - a_{14} a_{23})$ imply $b_5 = b_8 = 0$. Thus Δ_1 consists of three lines through M. For suitable λ, $S_2 + \lambda S_1$ is independent of x_3, so we have a subpencil with vertex X_3. There are thus no new nets in this case.

We now summarize the classification in this section.

THEOREM 1.7. *If we fix the singular subpencil $\nu = 0$ with plane-pair at* M, *and $\Delta = \nu\Delta_1$; then:*

Case $b_3 \neq 0$, $\delta \neq 0$: if ν meets Δ_1 in three distinct points, there is one net corresponding to each tangent from M *to Δ_1 which is not a component of Δ_1.*

Case $b_3 = 0$, $\delta \neq 0$: if ν meets Δ_1 twice at M *and in one further point, and* M *is a simple point of Δ_1, the same holds for Δ_1 irreducible; for Δ_1 reducible we obtain no net.*

If M *is a double point, there is one net for each tangent at* M, *which is not a component of Δ_1.*

Case $b_3 \neq 0$, $\delta = 0$: if ν meets Δ_1 once at M, *but is tangent elsewhere, there is just one net unless ν goes through a double point of Δ_1 when there is none.*

Case $b_3 = \delta = 0$: if ν meets Δ_1 three times at M, *there is one net for* M *a simple point; none for* M *a double point.*

1.6. *Unstable nets*

By theorem 0.1, for an unstable net we may choose coordinates x_i such that $a_{ij} = b_{ij} = c_{ij} = 0$ for $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant 4 - s$ where either $s = 1$ (type I) or $s = 2$ (type II). We begin by considering type II: nets of quadrics with a common line $x_3 = x_4 = 0$.

1.6.1. Consider the submatrix B formed from the first two rows and the last two columns. Changes of coordinates $x_1 : x_2$ and $x_3 : x_4$ have the effect of elementary row and column operations on B (changing x_1 and x_2 modulo x_3 and x_4 does not affect B). We may interpret the entries of B as lying in $\mathbb{C}^4$, with preferred quadric $\det B = 0$: we are allowed coordinate changes which fix this quadric and preserve the two systems of generators. As the entries are linear functions of $(\lambda, \mu, \nu) \in \mathbb{C}^3$, we have a map $\phi : \mathbb{C}^3 \to \mathbb{C}^4$, whose image may not lie in the quadric (else the net is not semistable). The image is thus (projectively) a plane (general or tangent), a line (chord or tangent) or a point not on the quadric. Thus we may change coordinates to reduce B to one of the forms $\begin{bmatrix} \nu & \lambda \\ \mu & \nu \end{bmatrix}$, $\begin{bmatrix} 0 & \lambda \\ \mu & \nu \end{bmatrix}$, $\begin{bmatrix} 0 & \lambda \\ \mu & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & \lambda \\ \lambda & \mu \end{bmatrix}$, $\begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}$. In all but the first, $a_{13} = b_{13} = c_{13} = 0$ so the net also belongs to type I; there is thus only the first to consider.

Δ a repeated conic. We add suitable multiples of x_3, x_4 to x_1, x_2 to normalize coordinates by $a_{34} = a_{44} = b_{33} = b_{34} = 0$, and so obtain

$$\begin{bmatrix} 0 & 0 & \nu & \lambda \\ 0 & 0 & \mu & \nu \\ \nu & \mu & a_{33}\lambda + c_{33}\nu & c_{34}\nu \\ \lambda & \nu & c_{34}\nu & b_{44}\mu + c_{44}\nu \end{bmatrix}.$$

On the singular conic $(\lambda, \mu, \nu) = (\theta^2, \phi^2, \theta\phi)$ we observe that the rank of this equals that of

$$\begin{bmatrix} 0 & \phi & \theta \\ \phi & a_{33}\theta^2 + c_{33}\theta\phi & c_{34}\theta\phi \\ \theta & c_{34}\theta\phi & c_{44}\theta\phi + b_{44}\phi^2 \end{bmatrix}$$

which has determinant

$$-a_{33}\theta^4 - c_{33}\theta^3\phi + 2c_{34}\theta^2\phi^2 - c_{44}\theta\phi^3 - b_{44}\phi^4.$$

Unless all the coefficients vanish, this has four roots, determining four points on the conic (which may coincide in various ways). The indication of these points determines a unique net. If all vanish, all points of the conic correspond to plane-pairs. This too gives a unique net. Here there is another common line $x_1 = x_2 = 0$, skew to the first.

1.6.2. We now consider nets of type I. Here we may take $a_{14}\lambda + b_{14}\mu + c_{14}\nu$ as the coordinate ν: it may not vanish, else the net is not semistable. Then Δ equals ν^2 multiplied by the discriminant of the 'net' N cut on the line $x_1 = x_4 = 0$. Observe that the original net, together with the flag $X_1 \in x_4$, determines N and the preferred subpencil P given by $\nu = 0$. We shall return to consider which nets have more than one such degenerate flag, and classify according to pairs (N, P) in the first instance.

(1) P a general pencil $\lambda x_2^2 + \mu x_3^2$.

There are two essentially distinct cases for N, with matrices

(i) $\begin{bmatrix} \lambda & \nu \\ \nu & \mu \end{bmatrix}$ or (ii) $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$; correspondingly Δ is a conic with repeated chord or a triangle with one side repeated.

(2) P a degenerate pencil $\lambda x_2^2 + 2\mu x_2 x_3$.

Again there are two cases (iii) $\begin{bmatrix} \lambda & \mu \\ \mu & \nu \end{bmatrix}$ and (iv) $\begin{bmatrix} \lambda & \mu \\ \mu & 0 \end{bmatrix}$: we have a conic with repeated tangent or a repeated line-pair.

(3) P reduces to a point-pair $2\lambda x_2 x_3$.

Here we have (v) $\begin{bmatrix} \nu & \lambda \\ \lambda & \nu \end{bmatrix}$, (vi) $\begin{bmatrix} \nu & \lambda \\ \lambda & 0 \end{bmatrix}$ and (vii) $\begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}$: three concurrent lines (one repeated) or (in two cases) a repeated line-pair.

(4) P reduces to a repeated point λx_2^2.

Since we are assuming the net semistable, we have just (viii) $\begin{bmatrix} \lambda & 0 \\ 0 & \nu \end{bmatrix}$ and (ix) $\begin{bmatrix} \lambda & \nu \\ \nu & 0 \end{bmatrix}$: Δ consists of two lines (multiplicities 1, 3) or a fourfold line.

(5) P vanishes identically:

(x) $\begin{bmatrix} 0 & \nu \\ \nu & 0 \end{bmatrix}$: Δ is a fourfold line.

The first three cases are those where Δ has no quadruple point. These admit investigation along the same lines as stable cases.

Cases 1(*i*), 1(*ii*): Δ a conic or line-pair with repeated chord. These cases are so similar we may treat them together. We have

$$\begin{bmatrix} 0 & 0 & 0 & \nu \\ 0 & \lambda & \epsilon\nu & a_{24}\lambda+b_{24}\mu+c_{24}\nu \\ 0 & \epsilon\nu & \mu & a_{34}\lambda+b_{34}\mu+c_{34}\nu \\ \nu & a_{24}\lambda+b_{24}\mu+c_{24}\nu & a_{34}\lambda+b_{34}\mu+c_{34}\nu & a_{44}\lambda+b_{44}\mu+c_{44}\nu \end{bmatrix},$$

where $\epsilon = 1$ for type I*a* and $\epsilon = 0$ for type I*c*. We normalize coordinates by choosing $x_2' = x_2+ux_4$, $x_3' = x_3+vx_4$ to make $a_{24} = b_{34} = 0$, and $x_1' = x_1+\alpha x_2+\beta x_3+\gamma x_4$ to make $c_{24} = c_{34} = c_{44} = 0$. Then for plane-pairs on $\nu = 0$ we have

$$0 = -a_{34}^2\lambda^3+a_{44}\lambda^2\mu+b_{44}\lambda\mu^2-b_{24}^2\mu^3.$$

In general we have three points (with multiplicites), the datum of which determines the remaining coefficients (up to signs and common multiples, easily adjusted by trivial coordinate changes). Again there is also the possibility that the remaining coefficients all vanish, and $\nu = 0$ yields a pencil of plane-pairs with axis $x_2 = x_3 = 0$: this net also is uniquely determined.

Case 2(*iii*) Δ a conic with repeated tangent. We normalize coordinates by $b_{24} = b_{34} = c_{24} = c_{34} = c_{44} = 0$, as above, thus obtaining

$$\begin{bmatrix} 0 & 0 & 0 & \nu \\ 0 & \lambda & \mu & a_{24}\lambda \\ 0 & \mu & \nu & a_{34}\lambda \\ \nu & a_{24}\lambda & a_{34}\lambda & a_{44}\lambda+b_{44}\mu \end{bmatrix}.$$

Then plane-pairs on $\nu = 0$ are given by

$$-a_{34}^2\lambda^3+2a_{24}a_{34}\lambda^2\mu-a_{44}\lambda\mu^2-b_{44}\mu^3 = 0.$$

In general we have three preferred points which (as before) determine the net. However, if $\mu = 0$ (the point of contact of the tangent) corresponds to a plane-pair, it then has multiplicity at least two, and the datum of the third point fails to yield information about the coefficient a_{24}. We shall see later how to account for this. Note that if $a_{44} = a_{24}^2$, the rank of S_1 drops to 1. If all points of $\nu = 0$ correspond to plane-pairs, we have two cases $a_{24} = 0$, $a_{24} \neq 0$: that the value of a_{24} here does not matter is seen by multiplying x_1 and x_4 by appropriate factors.

The cases remaining are those in which Δ has a quadruple point. It seems that Δ, even with indications of which points correspond to plane-pairs (and the finer information of this kind to be discussed in the next section) is inadequate in these cases to determine the $SL_4(\mathbb{C})$-orbit of the net. We will confine ourselves to a listing of nets up to equivalence under $GL_4(\mathbb{C})\times GL_3(\mathbb{C})$. Here we use a different approach.

1.6.3. Since $a_{14}\lambda+b_{14}\mu+c_{14}\nu = \nu$, we may normalize via $x_1' = x_1+\alpha x_2+\beta x_3+\nu x_4$ to make $c_{24} = c_{34} = c_{44} = 0$. All coefficients now appearing either contribute to the net N cut on

$$x_1 = x_4 = 0$$

or to the pencil Π with $\nu = 0$, in which x_1 does not appear, so we may consider it a pencil of conics in $x_1 = 0$. Moreover, a pair (N, Π) appears if and only if N and Π induce the same pencil P on $x_1 = x_4 = 0$. We have already classified pairs (N, P) up to isomorphism; we are about to classify pairs (Π, P). To infer a complete listing of triples, and hence of semistable nets, we need to know that an automorphism of P extends to one of N. Case by case checking shows that this is so, subject to qualification that in certain cases N induces some further geometry on the line $x_1 = x_4 = 0$, as follows:

Case (v): two points harmonically separating the point-pair $x_2 x_3$.

Case (vi): choice of one factor of $x_2 x_3$.

Case (viii): one point (x_3) other than x_2.

Case (x): a point-pair $(x_2 x_3)$.

Now we must consider Π. This is a pencil of plane conics, so is determined by a Segre symbol we have eight cases, five non-singular and three singular. The pencil P is cut on a line in the plane; we have case 1 for a general line, 2 for a line through a base point of Π, 3 for a line through two base points, 4 for a common tangent at a base point and 5 for a base line of the pencil. The ways these arise are illustrated in the following table.

type of pencil	[1,1,1]	[2,1]	[3]	[(1,1),1]	[(2,1)]	[;1;]	[1,1;;1]	[2;;1]
base point or description	*abcd*	*aabc*	*aaab*	*aabb*	*aaaa*	*xz*, *yz*	x^2, y^2	x^2, *xy*
lines of type 1	*ef*	*ef*	*ef*	*ef*	*ef*	—	*z*	—
2	*ae*	*ae*, *be*	*ae*, *be*	*ae*	*ae*	$x+z$	—	*z*
3	*ab*	*ab*, *bc*	*ab*	*ab*	—	*x*	—	—
4	—	*aa*	*aa*	*aa*	*aa*	—	*x*, $x+y$	*y*
5	—	—	—	—	—	*z*	—	*z*

In four places in the table there are two non-isomorphic cases, as indicated. Also, from the geometry on the line there are further bifurcations of cases as follows:

Case 3 (v): [1, 1, 1] has a special case when one member of the point-pair is the intersection of *ab* with *cd*.

Case 3 (vi): the six cases listed split into nine.

Case 4 (viii): types [2, 1] and [(1, 1), 1] have special cases when the chosen point lies on *bc* resp. *bb*.

Case 5 (x): the final type splits up according as the vertex of the pencil does or does not belong to the point-pair.

Note that if Π is not a genuine pencil, S_0 and S_1 are dependent so we do not have a genuine net.

Note also, for comparison of our two approaches, that the discriminant of Π is precisely the equation for plane-pairs in the singular pencil $\nu = 0$.

We must now consider the possibility that the flag $X_1 \in x_4$ is not determined uniquely by the net. Suppose there are two such flags. If both have the same vertex, or both the same plane, then the net is not semistable (it has a common vertex resp. a degenerate flag defined by a line in the plane). If neither point lies in the other plane, we may take the flags as $X_1 \in x_4$, $X_4 \in x_1$. Here the net is determined by N, which must be of type 1 or 2 (otherwise we only have a pencil); P appears as z in the above table. If just one point lies in the other plane, we may take the flags as $X_1 \in x_4$, $X_2 \in x_1$. In this case, X_1 is a common vertex for the net, which cannot then be semistable.

Finally, if each point lies in the other plane we may take the flags as $X_1 \in x_4$, $X_2 \in x_3$ and have

$$\begin{bmatrix} 0 & 0 & 0 & \nu \\ 0 & 0 & \rho & 0 \\ 0 & \rho & ? & ? \\ \nu & 0 & ? & ? \end{bmatrix}.$$

If ρ is a multiple of ν, we may take $\rho = \nu$ and then see that (N, P) is of type 4(ix) or 5(x) and Π of type [11;;1] or [2;;1]: interchanging the flags shows that the final entries in the 4th and 5th lines of our table yield equivalent nets. If ρ is not a multiple of ν, we can take it as μ. We then normalize coordinates to put the remaining terms of the matrix in the form

$$\begin{bmatrix} a_{33}\lambda + c_{33}\nu & a_{34}\lambda \\ a_{34}\lambda & a_{44}\lambda + b_{44}\mu \end{bmatrix}.$$

Cases may be tabulated as follows.

For (N, P) we have 2(iv) if $a_{33} \neq 0$, 3(vi) if $a_{33} = 0$, $c_{33} \neq 0$, 3(vii) if $a_{33} = c_{33} = 0$.

For Π, if $a_{33}a_{44} \neq a_{34}^2$, [2, 1] if $a_{44} \neq 0$, [3] if $a_{44} = 0$, $b_{44} \neq 0$, [;1;] if $a_{44} = b_{44} = 0$ if $a_{33}a_{44} = a_{34}^2$, [(1,1), 1] if $a_{44} \neq 0$, [(2, 1)] if $a_{44} = 0$, $b_{44} \neq 0$, [2;;1] if $a_{44} = b_{44} = 0$.

If $a_{33}a_{44} \neq a_{34}^2$ (and similarly if equality holds) we thus have 9 cases which may be arranged as a matrix; interchanging the two flags has the effect of transposing the matrix, so the number of different cases is reduced by 6.

In this final case, the quadrics have a common line as well as two types of singular flag. In general, we have a common line in cases (iv), (vi), (ix) and two intersecting common lines in cases (vii), (x).

1.6.4. *Non-semistable nets*

Here, if we write $u_{ij} = a_{ij}\lambda + b_{ij}\mu + c_{ij}\nu$, we may take $u_{ij} = 0$ for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant 5-r$. If $r = 1$, the equations are independent of x_1 (the quadrics have a common vertex), so the classification is the same as that for nets of conics.

Now suppose $r = 2$, If u_{14} and u_{24} are linearly dependent, we may replace x_1, x_2 by suitable linear combinations to reduce u_{14} to zero, and are thus in the above case; so we may suppose them independent. Now if u_{33} is dependent on u_{14}, u_{24}, but is non-zero, we may adjust $x_1 : x_2$ again and suppose $u_{14} = u_{33} = \lambda$, $u_{24} = \mu$. Now if $c_{34} \neq 0$, we normalize $x_3' = x_3 + \alpha x_4$ to make $c_{44} = 0$. Then altering x_1 and x_2 mod x_3 and x_4, we may suppose $a_{34} = b_{34} = a_{44} = b_{44} = 0$. We thus have two cases:

$$\lambda(2x_1x_4 + x_3^2) + \mu(2x_2x_4) + \nu(2x_3x_4 \text{ or } x_4^2).$$

If $u_{33} = 0$ we argue similarly; here $c_{34} \neq 0$ else the net would have a common vertex X_3. We thus have

$$2x_4(\lambda x_1 + \mu x_2 + \nu x_3).$$

Finally, if u_{33} is independent of u_{14}, u_{24} we may take it as ν (and $u_{14} = \lambda$, $u_{24} = \mu$). Again choose α to make $c_{34} = 0$ and normalize $a_{34} = b_{34} = a_{44} = b_{44} = 0$. We obtain two final cases

$$\lambda 2x_1x_4 + \mu 2x_2x_4 + \nu(x_3^2 \text{ or } x_3^2 + x_4^2).$$

As already stated, we regard the unstable nets as of less interest. We shall not attempt a summary of the conclusions of this section.

2. Base points of the adjugate system

2.1. *General remarks*

The enumeration in §1 has the appearance of being somewhat haphazard. In this part we pave the way for the more theoretical considerations at the end of the paper by going over the listing more systematically and making the numerical results explicit.

Given a net $\lambda S_0 + \mu S_1 + \nu S_2$, where $S_i = \boldsymbol{x}^{\mathrm{T}} A_i \boldsymbol{x}$, A_i a symmetric $n \times n$ matrix and $\boldsymbol{x}^{\mathrm{T}} = (x_1, \ldots, x_n)$, the dual equation of the general quadric of the system is given by

$$\boldsymbol{X}^{\mathrm{T}} \operatorname{adj} (\lambda A_0 + \mu A_1 + \nu A_2)\, \boldsymbol{X} = 0,$$

where adj denotes the adjugate matrix (we recall the identity $A \operatorname{adj} A = (\det A)\, I$; the entries in adj A are the $(n-1) \times (n-1)$ minors of those in A, up to sign), and $\boldsymbol{X}^{\mathrm{T}}$ is the row matrix $(X_1, \ldots, X_n)$. Instead of regarding this as an equation for $\boldsymbol{X}$, we take $\boldsymbol{X}$ as parameter and regard it as an equation for $(\lambda : \mu : \nu)$. We thus have (in general) an ∞^{n-1} system of curves of degree $(n-1)$: this we call the *adjugate system* for the net. The advantage is that this is independent of the choice of coordinates $\boldsymbol{x}$.

The adjugate net was introduced (for $n = 4$) in the classical paper of Hesse (1855*a*), for essentially the present purpose. It was shown by Dixon (1902) that a general net can be recovered (up to choice of coordinates) from the adjugate system. Dixon's technique is exploited by Edge (1938) and on p. 161 of Edge (1947). The essential point to note here is that $\operatorname{adj} \operatorname{adj} A = (\det A)^{n-2} A$.

Our proof of the following characteristic property of the adjugate system is essentially that of Hesse (1855*a*, p. 294).

Lemma 2.1. *Any curve Γ of the adjugate system intersects Δ in a set of points each with even multiplicity.*

Proof. We may take coordinates so that Γ corresponds to $\boldsymbol{X} = (0, \ldots, 0, 1)$ and so is given by the vanishing of the principal minor of $\lambda A_0 + \mu A_1 + \nu A_2 = A$, say. Recall the determinantal identity:

$$(\text{cofactor of } a_{11} a_{22} - a_{12}^2) \det A = (\text{cofactor } a_{11})\,(\text{cofactor } a_{22}) - (\text{cofactor } a_{12})^2.$$

For a general net, the adjugate curves corresponding to cofactor a_{11} (i.e. to x_1) and to x_2 will intersect Δ in disjoint sets of points; the equation shows that the union of these sets is a set of points of even multiplicity, whence the assertion in this case. But now the multiplicities must remain even under specialization.

Remark. It may be objected that the equation of Γ could reduce to $0 = 0$. We have observed that for a semistable net, this will not be the case for a general Γ. In fact, there exist Γ of this type if and only if the net is unstable. For Γ corresponding to $x_n = 0$ so collapses if and only if the trace of the net on $x_n = 0$ is singular. By the characterization theorem in §1 (with $n-1$ in place of n), we can find r, $1 \leqslant r \leqslant \frac{1}{2}n$ and coordinates x_i so that $a^i_{jk} = 0$ whenever $j \leqslant r$, $k \leqslant n-r$. But then by the other part of the theorem, the net fails to be stable. The converse follows by reversing the argument.

Although for a general net the adjugate system has no base points, in special cases it may well have. We adopt these as a systematic principle in listing the conclusions of §1. To facilitate the determination of these base-points, we begin with some further lemmas.

Lemma 2.2. N *is a base point of the adjugate system if and only if rank $S_2 \leqslant n-2$.*

Proof. We seek the condition that $\boldsymbol{X}^{\mathrm{T}} \operatorname{adj} (\lambda A_0 + \mu A_1 + \nu A_2)\, \boldsymbol{X} = 0$ for all $\boldsymbol{X}$ at $(\lambda, \mu, \nu) = (0, 0, 1)$; i.e. that adj A_2 vanish identically. This is equivalent to rank $A_2 \leqslant n-2$.

While this result is already sufficient to deal with ordinary singularities of Δ, we find that when higher singularities occur, it is possible to have 'implicit' (or 'infinitely near') base points. We shall deal with most of these *ad hoc*, but it will save time to insert a further general result here.

Lemma 2.3. (i) *If rank* $S_2 = n-j-1$, *each member of the adjugate system has a j-fold point at* N.

(ii) *If rank* $S_2 = n-2$, *then the members of the adjugate system all touch* MN *at* N *if and only if* S_1 *contains the vertex of* S_2 (*a projective line*).

Proof. (i) We may take the matrix of S_2 in standard form: then in $\lambda S_0+\mu S_1+\nu S_2$, ν only appears in the first $(n-j-1)$ rows. It is thus clear that, for any $\boldsymbol{X}$, the degree of $\boldsymbol{X}^{\mathrm{T}}\operatorname{adj}(\lambda S_0+\mu S_1+\nu S_2)\,\boldsymbol{X}$ in ν is at most $n-1-j$, so each term has degree at least j in λ and μ, which yields the stated result.

(ii) Continuing from the above, with $j = 1$, the terms of degree precisely 1 in λ and μ, $(n-2)$ in ν are

$$\nu^{n-2}((a^1_{n,n}\lambda+a^2_{n,n}\mu)\,X^2_{n-1}-2(a^1_{n-1,n}\lambda+a^2_{n-1,n}\mu)\,X_{n-1}X_n+(a^1_{n-1,n-1}\lambda+a^2_{n-1,n-1}\mu)\,X^2_n),$$

so Δ touches $\lambda = 0$ at $(0,0,1)$ when

$$a^2_{n,n}X^2_{n-1}-2a^2_{n-1,n}X_{n-1}X_n+a^2_{n-1,n-1}X^2_n = 0.$$

This happens for all $\boldsymbol{X}$ if and only if $a^2_{n,n} = a^2_{n-1,n} = a^2_{n-1,n-1} = 0$, which expresses the stated condition.

In case (ii), we may go rather further.

Lemma 2.4. *If the adjugate system has coincident base points at* N, *in the direction* MN, *then the coefficient of* $\nu^{n-3}\mu^3$ *in* Δ *vanishes. Thus either* N *has multiplicity greater than* 2 *on* Δ *or it is a double point of type* A_k *with* $k \geqslant 3$.

Proof. In view of the form of S_2, the desired coefficient is

$$\sum_{i=1}^{n-2}\det\begin{bmatrix} a^2_{i,i} & a^2_{i,n-1} & a^2_{i,n} \\ a^2_{i,n-1} & a^2_{n-1,n-1} & a^2_{n-1,n} \\ a^2_{i,n} & a^2_{n-1,n} & a^2_{n,n}\end{bmatrix} = 0 \quad \text{since} \quad a^2_{n-1,n-1} = a^2_{n-1,n} = a^2_{n,n} = 0.$$

Clearly it would be easy to obtain further results along these lines, but the complete pattern remains obscure.

2.2. *Multiplicities: a working hypothesis*

The other feature of our approach is a systematic calculation of multiplicities. This serves as a valuable check on the enumerations of §1. Recall that if Δ has no repeated component, there is a finite number of corresponding nets, and if multiplicities are counted correctly, this number must stay constant. It is not hard to see that it is 3 for $n = 3$ and 36 for $n = 4$, as was shown already by Hesse (1855*b*, p. 318). For a fuller discussion see the final section.

We begin by going over the case $n = 3$. For Δ irreducible, we normalized

$$\Delta = -\mu^2\nu+\lambda^3+p\lambda\nu^2+q\nu^3,$$

and found one net for each root of $\lambda^3+p\lambda+q = 0$. Thus for Δ non-singular there are three corresponding nets (called of type A), and the adjugate systems cannot have base points. If Δ is nodal, we may choose the repeated root (type B, multiplicity 2) or the other (type B*, multiplicity 1) The conic corresponding to the node N is a repeated line for type B, a line-pair for type B*: thus N is a base point of the adjugate system for type B, but not for type B*. Finally, if Δ is cuspidal we have a single net (type C, multiplicity 3), and the cusp is necessarily a base point of the adjugate system.

The key observation to be drawn from the listings of § 1 – though we shall study much finer points also – is that the listing of cases depended only on the number and type of singular points of Δ or, more precisely, on the equi-singularity class of Δ. For example, dual singularities (and such questions as whether an inflexion occurs at a node) are not relevant. Now as we vary Δ in such a class the number of nets remains constant, so their multiplicities must remain constant too.

It follows from (2.1) to (2.4) that the families used in § 1 for the classification correspond to the possibilities of base points for the adjugate system. Observing that the multiplicity exceeds 1 precisely when the adjugate system has a base point, we seek a connection in other cases.

Working hypothesis. If Δ is irreducible, the multiplicity of any net with discriminant Δ is determined by the base points of the adjugate system; more precisely, by their equi-singularity class. Moreover, for separated singularities, we simply multiply the corresponding multiplicities.

From the above, a double point with distinct tangents (node) corresponds to multiplicity 2, a cusp to multiplicity 3. We now begin our detailed enumeration for the case $n = 4$.

2.3. Δ *irreducible with double points only*

First suppose Δ has ordinary singularities only. Then Δ may only have nodes (A_1) or cusps (A_2), and as Δ is irreducible it may have at most 3 such. When there are 3, the results of our calculations are listed in theorem 1.3. Since there are no implicit base-points by lemma 2.4 and explicit ones occur (lemma 2.2) precisely for plane-pairs in the net, all base points are determined. We obtain table 1, where the entry $r \times m$ denotes r nets, each of multiplicity m.

TABLE 1

base points … singularities	—	L	M	N	L, M	L, N	M, N	L, M, N
A_1(L, M, N)	4×1	2×2	2×2	2×2	1×4	1×4	1×4	1×8
A_2(L) A_1(M, N)	2×1	2×3	1×2	1×2	1×6	1×6	0×4	1×12
A_2(L, M) A_1(N)	1×1	1×3	1×3	1×2	1×9	0×6	0×6	1×18
A_2(L, M, N)	0×1	1×3	1×3	1×3	0×9	0×9	0×9	1×27

The total multiplicity adds up (as it should) to 36 in each case.

More significantly, for each node A (resp. cusp B) the total multiplicity of nets with adjugate base point at A (resp. B) is 20 (resp. 30). As it seems unreasonable that, in a deformation, this could change unless the nature of A (resp. B) itself changes during a deformation, we adopt this 'principle of permanence' as a second working hypothesis for determining multiplicities.

As a first consequence, we can determine what happens in the remaining cases: see table 2.

TABLE 2

Nonsingular: 36×1
one node: 16×1 (no base point) + 10×2 (one)
one cusp: 6×1 (no base point) + 10×3 (one)
and for two,

	—	L	M	L, M
A_1(L, M)	8×1	4×2	4×2	3×4
A_2(L) A_1(M)	4×1	4×3	1×2	3×6
A_2(L, M)	3×1	1×3	1×3	3×9

The appearance of 10 nets above confirms the observations made after lemma 1.1.

Now we consider the case when Δ has a higher double point. Recall that we considered three cases here, according to the type of the pencil tangent at the double point N. From lemmas 2.2

and 2.4, N is not a base point for type [4]; it is a simple base point for type [(3,1)], and is a base point of multiplicity at least 2 for pencils of type [(2,2)]. Next we must fix the multiplicities for this case. We have

$$A = \begin{bmatrix} a_{11}\lambda + b_{11}\mu & \nu & b_{13}\mu & b_{14}\mu \\ \nu & a_{22}\lambda + b_{22}\mu & b_{23}\mu & b_{24}\mu \\ b_{13}\mu & b_{23}\mu & 0 & \lambda \\ b_{14}\mu & b_{24}\mu & \lambda & 0 \end{bmatrix},$$

$$\Delta = \det A = \{\lambda\nu - (b_{14}b_{23} + b_{13}b_{24})\,\mu^2\}^2 - \{a_{11}\lambda^2 + b_{11}\lambda\mu - 2b_{13}b_{14}\mu^2\}\{a_{22}\lambda^2 + b_{22}\lambda\mu - 2b_{23}b_{24}\mu^2\}.$$

Now $\boldsymbol{X}^{\mathrm{T}}\operatorname{adj} A\boldsymbol{X}$ has tangent $\lambda = 0$ at N; for the second-order behaviour we need the coefficients of $\lambda\nu^2$ and of $\mu^2\nu$ which are respectively $2X_3X_4$ and $2(b_{14}b_{24}X_3^2 - (b_{14}b_{23} + b_{13}b_{24})\,X_3X_4 + b_{13}b_{23}X_4^2)$. The ratio of these is constant if and only if $b_{14}b_{24} = b_{13}b_{23} = 0$: if this fails, there are no further implicit adjugate base points. If it holds, the second neighbourhood points on $\boldsymbol{X}^{\mathrm{T}}\operatorname{adj} A\boldsymbol{X}$, on the conic $\lambda\nu = (b_{14}b_{23} + b_{13}b_{24})\,\mu^2$ and on Δ coincide.

As Δ is irreducible, and so not divisible by λ, the coefficient $(b_{13}b_{24} - b_{14}b_{23})^2$ of μ^4 does not vanish. For the higher base point $b_{14}b_{24} = 0$: we may suppose by symmetry $b_{14} = 0$. Then $b_{13}b_{24} \neq 0$, so we obtain the extra base point if and only if $b_{23} = 0$. This is equivalent to saying that each of the preferred pairs of factors of the quartic includes λ: it implies λ a repeated root (type A_5 or A_6); and for A_5 we have two nets, one with this base point and one without; for A_6 just one net, with. I shall omit the proof that higher base points do not occur.

The results are presented in table 3. Denote the more complex double point by L, the simpler (if there is one) by M.

TABLE 3

base points ...	—	L	L, L	M	L, M	L, L, M
singularities						
A_3	2 × 1	4 × 4	3 × 6			
A_4	1 × 1	1 × 5	3 × 10			
A_3+A_1	2 × 1	2 × 4	1 × 6	0 × 2	1 × 8	1 × 12
A_3+A_2	2 × 1	1 × 4	0 × 6	0 × 3	1 × 12	1 × 18
A_4+A_1	1 × 1	1 × 5	1 × 10	0 × 2	0 × 10	1 × 20
A_4+A_2	1 × 1	1 × 5	0 × 10	0 × 3	0 × 15	1 × 30

base points ...	—	L	L, L	L, L, L
A_5	1 × 1	0 × 6	1 × 15	1 × 20
A_6	1 × 1	0 × 7	0 × 21	1 × 35

The multiplicities for A_3, A_4 are easily determined from the working hypotheses above: for example, in the case of A_4 let adjugate base points L resp. L, L correspond to multiplicities p resp. q. Then for A_4+A_1 the only net with M a base point has multiplicity 20 by the principle of permanence, and this equals $2q$.

There is not information yet to determine the multiplicities for A_5 and A_6, though the final one in the table must be 35. However, if we assemble the table of multiplicities so far obtained

	—	L	L, L	L, L, L
A_1	1	2		
A_2	1	3		
A_3	1	4	6	
A_4	1	5	10	
A_5	1	?	?	?
A_6	1	?	?	35

the following is extremely plausible (and, for A_5, consistent):

Conjecture 1. *For r coincident base points at a singular point of Δ of type* A_n, *we have*

$$\text{(i) } r \leqslant \frac{n+1}{2} \qquad \text{(ii) } \textit{multiplicity} = \binom{n+1}{r}.$$

2.4. *Δ reducible with double points only*

The multiplicites appropriate to the double points must be as above; but that some modification is necessary can be seen by considering the case of nets of conics. If Δ is a conic with chord there are two singular points, each of type A_1 and two nets, one with no adjugate base point and multiplicity 1, and one with two adjugate base points, but multiplicity 2 rather than 4. Similarly if Δ is a conic with tangent, the singular point has type A_3, and the unique net (type F*) has double adjugate base point there, but the multiplicity is 3 rather than 6. In the remaining case when Δ is a triangle, so we have 3 singular points each of type A_1, there is one net with no adjugate base points and multiplicity 1, and one with 3 adjugate base points, but multiplicity 2 rather than 8.

The following rule gives these values correctly, and gives consistent results for nets of quadrics also.

Conjecture 2. *If Δ is reducible, but has only double points, and various (ordinary and implicit) adjugate base points* P_i *are prescribed, then blow up Δ at all the* P_i. *If the result has k components, the multiplicity is* 2^{1-k} *times that predicted by a naïve intrepretation of conjecture* 1.

Observe that conjecture 1 (i) states that simple points cannot be base points: this would correspond to superfluous blowing-up. Observe also that – as is already evident in the above case of nets of conics – simple numerical criteria restrict the choices of sets of base points.

If a line ν is a component of Δ, a curve of degree $(n-1)$ meets it in $(n-1)$ points. If these are to coincide in pairs, other than those attributed to adjugate base points, where another component of Δ also appears, then the number of adjugate base points on ν must have the opposite parity to n. Similarly if $n = 4$ and δ is a conic contained in Δ, we must have an even number of base points on δ.

Thus, for example, if Δ consists of two conics with four distinct intersections, either 0, 2 or 4 of these correspond to plane-pairs (rather than cones): a conclusion obtained in theorem 1.3 as a consequence of the enumeration.

We return to the enumeration. In the case of a pencil ν with common vertex, we see directly that adjugate base points are precisely those of the net of conics, together with the intersection points $\nu \cap \Delta_1$, counted with appropriate multiplicities. (As Δ has no triple points, ν passes through no double point of Δ_1.)

Now let $\Delta = \nu\Delta_1$, where ν does not correspond to a pencil with common vertex; and let M be the point on ν corresponding to a plane-pair. Then M is an adjugate base point: I claim that, even if ν touches Δ_1 at M, the neighbouring point is not. For take $\boldsymbol{X} = (0, 0, 1, 0)$. The corresponding member of the adjugate system is of the form $\nu^2 f_1(\lambda, \mu, \nu) - (\lambda + a_{11}\nu)(\mu + a_{24}\nu)^2$, which never contains the point in question.

As to other base points, we will refer to lemma 1.2. Since we have a singular subpencil

$$S_0 = x_1^2 + 2x_2x_3, \quad S_1 = 2x_2x_4,$$

the condition for $\lambda S_0 + \nu S_2$ to meet $x_2 = 0$ in a degenerate conic is

$$0 = \det \begin{bmatrix} \lambda + a_{11}\nu & a_{13}\nu & a_{14}\nu \\ a_{13}\nu & a_{33}\nu & a_{34}\nu \\ a_{14}\nu & a_{34}\nu & a_{44}\nu \end{bmatrix} = \delta\lambda\nu^2 + \nu^3(a_{11}a_{33}a_{44} + \text{etc.}),$$

so the corresponding tangents are $\nu = 0$ (twice) and the one named as preferred. In the case $\delta = 0$ we have $\nu = 0$ as threefold root. (Note that S_0 does not meet $x_4 = 0$ in a degenerate conic, so $\nu = 0$ does not belong to the other set.) The results can now be read off.

We start tabulation by adjugate base points in the case when $\Delta = \nu\Delta_1$ with Δ_1 an irreducible cubic. In table 4, the nature of Δ_1 is indicated by the letter E, N or C according as Δ_1 is elliptic, nodal or cuspidal: in the latter two cases the double point is denoted by P. As we are excluding triple points, the line $\nu = 0$ does not pass through P; the nature of its intersection with Δ_1 is indicated by the letter *c* (chord through the points L, M, N), *t* (tangent at L, cutting Δ_1 in M) or *f* (inflexional tangent at L). Recall that the number of base points on ν is odd: 3 if the corresponding pencil has a common vertex and 1 if not.

TABLE 4

Δ_1	ν	base points ... singularities	L	M	N	LMN	LP	MP	NP	LMNP
E	c	$3A_1$	4×2	4×2	4×2	3×4				
N	c	$4A_1$	2×2	2×2	2×2	1×4	1×4	1×4	1×4	1×8
C	c	$3A_1+A_2$	1×2	1×2	1×2	0×4	1×6	1×6	1×6	1×12

Δ_1	ν	singularities / base points ...	L	M	LLM	LP	MP	LLMP
E	t	A_3+A_1	4×4	1×2	3×6			
N	t	A_3+2A_1	2×4	1×2	1×6	1×8	0×4	1×12
C	t	$A_3+A_2+A_1$	1×4	1×2	0×6	1×12	0×6	1×18

Δ_1	ν	singularities / base points ...	L	LLL	LP	LLLP
E	f	A_5	1×6	3×10		
N	f	A_5+A_1	1×6	1×10	0×12	1×20
C	f	A_5+A_2	1×6	0×10	0×18	1×30

Next we list the cases when Δ breaks up into two distinct non-singular conics. A point of multiplicity r as an intersection produces a singular point A_{2r-1}. Since the total intersection multiplicity is 4, cases are as listed in table 5.

TABLE 5

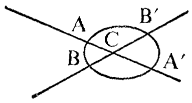
singularities								
	—	A, B	A, C	A, D	B, C	B, D	C, D	A, B, C, D
$4A_1$	4×1	1×4	1×4	1×4	1×4	1×4	1×4	1×8
	—	A, A	A, B	A, C	B, C	A, A, B, C		
A_3+2A_1	2×1	1×6	1×8	1×8	0×4	1×12		
	—	A, A	A, B	B, B	A, A, B, B			
$2A_3$	2×1	0×6	1×16	0×6	1×18			
	—	A, A	A, B	A, A, A, B				
A_5+A_1	1×1	1×15	0×12	1×20				
	—	A, A	A, A, A, A					
A_7	1×1	0×28	1×35					

Strictly speaking, we have not verified the precise multiplicity of A as adjugate base point for the two final higher cases: this is, however, straightforward.

We complete the tabulation with double points by listing cases where Δ splits into conic plus two lines, or four lines. We still have an odd number of base points on each line; an even number on each conic. The cases are given in table 6.

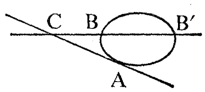
TABLE 6

$5A_1$



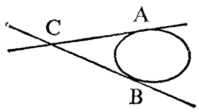
C	A, B	A, B′	A′, B	A′, B′	A, A′, C	B, B′, C	A, A′, B, B′, C
2×2	1×4	1×4	1×4	1×4	1×4	1×4	1×8

A_3+3A_1



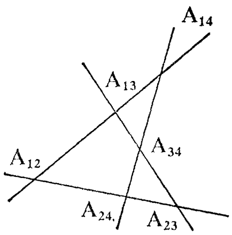
C	A, B	A, B′	B, B′, C	A, A, C	A, A′, B, B, C
1×2	1×8	1×8	0×4	1×6	1×12

$2A_3+A_1$



C	A, B	A, A, C	B, B, C	A, A, B, B, C
1×2	1×16	0×6	0×6	1×18

$6A_1$



(A_{12}, A_{34}) or (A_{13}, A_{24}) or (A_{14}, A_{23}) 1×4
(A_{12}, A_{13}, A_{14}) or (A_{12}, A_{23}, A_{24}) or (A_{13}, A_{23}, A_{34}) or (A_{14}, A_{24}, A_{34}) 1×4
and, with basepoints $A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34}$: 1×8.

Again the enumeration comes at once from our results on nets with a singular pencil. The results as a whole form a very thorough check both on those enumerations and on our conjectures concerning multiplicity. For the case when Δ has only double points, I feel certain that these are correct.

2.5. Δ *with a triple point*

New considerations enter with triple points, and the evidence for any conjectures is much thinner. First, consider a triple point N of type D_4 (i.e. with distinct tangents). We cannot simply impose N as an adjugate base point, since this corresponds to intersection multiplicity 3 (which is odd). One might think of a parity rule, but what we find in the examples below is that either (*a*) we impose with N a satellite base point in one of the tangent directions, or (*b*) we impose N as a double point on members of the adjugate system. Thus the most naïve extension of conjecture 1(i): that we can only impose satellite base points at non-simple points: is false. For the higher triple points, it is even less clear what to do, so our summary is of necessity more tentative. We may observe in general that if S_2 has rank $n-s$, not only is N an s-fold point of Δ, but it is an $(s-1)$-fold point of the adjugate system (which thus consists of sub-adjoints to Δ).

There are four types of Δ with D_4-singularity:

Irreducible. N a plane-pair: one net for each tangent direction. N a repeated plane: one net.

Nodal cubic plus nodal chord. If the pencil has common vertex, two nets (one for net of conics type B, one for type B*); otherwise two nets, corresponding to nodal tangents.

Conic plus two chords meeting on it. Here either or both may have common vertex; or neither, with one net in each case.

Four lines, of which three concur. Either just one of the concurrent lines corresponds to a pencil with common vertex, or all four lines do.

For each of these cases we have four nets, of which one each may be associated to the three tangents at N while N^2 is imposed for the fourth. Our multiplicity rules suggest the same results in each case; the principle of permanence now yields:

D_4 point plus one tangent direction: multiplicity 8,

D_4 point as double point: multiplicity 12.

For a D_5 singularity, Δ may be irreducible or a cuspidal cubic plus a chord through the cusp. In each case we obtain two nets: one corresponding to choice of the repeated tangent direction and one to imposing it as a double point. In the reducible case, only the latter has base point at the node. We conclude the corresponding multiplicities are 16 and 20.

For a D_6 singularity, Δ may be a nodal cubic plus nodal tangent or a conic plus chord plus tangent at one end of the chord. Just as for D_5, we get two nets each time, and make the same conclusion concerning multiplicities. If conjecture 2 applied in the presence of triple points (and we have no real evidence to support this) we should double these numbers to obtain the contribution of D_6 itself.

For E_6 we must have Δ irreducible. There is just one net (hence multiplicity 36); N corresponds to a repeated plane, so the adjugate system passes doubly through N. I have not checked for further implicit base points.

Finally, E_7 occurs when Δ consists of a cuspidal cubic plus the tangent at the cusp. There is just one net again, with N a repeated plane. Perhaps imposing E_7 as a double base point corresponds to multiplicity 72. There are no further singularity types for plane quartics without repeated components, except $\tilde{E}_7$: a quadruple point with distinct tangents, which appears when Δ consists of 4 concurrent lines. There is a unique corresponding type of 'net': taking N as the quadruple point, S_2 must vanish identically, so the net really reduces to a pencil (of type [1, 1, 1, 1]). This is too degenerate to justify further elaboration (though a direct proof of uniqueness here was the basis of my first proof that multiplicity stays finite for Δ with no repeated component, the corresponding result breaks down for $n \geqslant 5$, so the argument now presented is superior).

We conclude this section by clearing up the final case of 1.6.2. Since $a_{34} = 0$, the adjugate system is

$$X_1^2\{(\lambda\nu-\mu^2)(a_{44}\lambda+b_{44}\mu)-a_{24}^2\lambda^2\nu\}+2a_{24}\lambda\nu^2X_1X_2-2a_{24}\lambda\mu\nu X_1X_3 \\ -2\nu(\lambda\nu-\mu^2)X_1X_4-\nu^3X_2^2+2\mu\nu^2X_2X_3-\lambda\nu^2X_3^2.$$

The coefficient of λ^2 is $(a_{44}-a_{24}^2)\,\nu X_1^2$, so we have coincident base points at L along $\nu = 0$. For second-order base points we look at the coefficient of $\lambda\mu^2$: $-a_{44}X_1^2$. There is thus a third base point, along the conic $(a_{44}-a_{24}^2)\,\lambda\nu-a_{44}\mu^2 = 0$; and this determines the ratio $a_{44}: a_{24}^2$. If $a_{24} \neq 0$, there are no further base points.

3. Jacobian curves

3.1. *The case when Δ is non-singular*

I now recall the classical enumeration (Dixon 1902) of nets with Δ non-singular. I am grateful to Michael Atiyah for a conversation drawing my attention to this reference, which had a decisive impact on my approach to this problem. We return to the case of nets in $\mathbb{C}^n$, with n arbitrary.

The system of plane $(n-1)$ic curves cuts Δ in sets of $n(n-1)$ points forming a linear system λ which contains the sum of the canonical system κ with double the system η of linear sections. Each net $\mathscr{X}$ gives an adjugate system, contained in the above, which cuts Δ in a set of $\frac{1}{2}n(n-1)$ points (each of multiplicity 2) forming a system $\xi(\mathscr{X})$ on Δ whose double is λ. It is shown by Dixon (1902) that this system $\xi(\mathscr{X})$ determines $\mathscr{X}$. Conversely, suppose we are given a linear system ξ with $2\xi = \lambda$. We can interpret both λ and ξ as line bundles over Δ: then the additive notation becomes multiplicative, $\xi \otimes \xi \simeq \lambda$. Write $\Gamma(\xi)$ for the space of sections of ξ, and $\epsilon(\xi) = \dim \Gamma(\xi)$ mod 2. Then (see for example, Mumford 1971) $\epsilon(\xi)$ is stable under deformations and it can be shown that ξ corresponds to a net if and only if $\epsilon(\xi) = 0$.

The set of ζ with $\zeta \otimes \zeta \simeq \kappa$ is called the set of *theta-characteristics* $S(\Delta)$ of Δ. It corresponds bijectively by $\xi = \zeta \otimes \eta$ to the above set of systems ξ. Each is a principal homogeneous space over $J_2(\Delta)$, the group of line-bundles whose square is trivial: i.e. elements of order 2 in the Jacobian (or Picard) variety $J(\Delta)$, which may be canonically identified with $H^1(\Delta; \mathbb{F}_2)$. As Δ is a general curve of degree n, its genus is given by $p = \frac{1}{2}(n-1)(n-2)$, and $J_2(\Delta)$ can be considered an $\mathbb{F}_2$-vector space of dimension $2p$.

Mumford goes on to prove the result (attributed to Riemann) that ϵ is a quadratic function associated to the bilinear form of cup products on $H^1(\Delta; \mathbb{F}_2)$ – i.e. that for $\xi \in S(\Delta)$ and $\alpha, \beta \in H^1(\Delta; \mathbb{F}_2)$,

$$\epsilon(\xi+\alpha+\beta)+\epsilon(\xi+\alpha)+\epsilon(\xi+\beta)+\epsilon(\xi) \equiv \alpha\beta \pmod 2.$$

According to the theory of quadratic forms over $\mathbb{F}_2$, this situation is described by an invariant due to (Arf 1940), according to whose value the number of ξ with $\epsilon(\xi) = 0$ is $2^{2p-1} \pm 2^{p-1}$. Mumford concludes by showing that the positive sign must be taken. Thus the total number of nets in $\mathbb{C}^n$ with given Δ is $N = 2^{2p-1}+2^{p-1}$ for any non-singular Δ, and hence for any Δ free of repeated components. For $n = 3, 4, 5$ we have $p = 1, 3, 6$ hence $N = 3, 36, 2080$.

3.2. *Generalized jacobians*

It seems reasonable to expect some analogue of this theory to hold for singular Δ, and the remainder of this paper is devoted to formulating necessary properties of such an analogue.

At first sight there appear to be two alternative approaches: should we consider points of order 2 on the jacobian of the normalized curve, or should we use the generalized jacobians of (Rosenlicht 1954)? But it does not take much thought, on the basis of § 2, to see that we need both. More precisely, we now formulate

CONJECTURE 3. *For any (permissible) system B of adjugate base points on Δ, form a curve Δ_B by blowing up Δ along B (i.e. the transform of Δ by the system of curves of sufficiently high degree with base points B). Form the set $S(\Delta_B)$ of square roots of the canonical bundle of Δ_B: this is a principal homogeneous space over the group $J_2(\Delta_B)$ of elements of order 2 in the Jacobian, which is canonically isomorphic to $H^1(\Delta_B; \mathbb{F}_2)$. Then there is a quadratic form $\epsilon: S(\Delta_B) \to \mathbb{F}_2$, associated to the form defined by cup products on $H^1(\Delta_B; \mathbb{F}_2)$. For $\zeta \in S(\Delta_B)$, the system of curves meeting Δ at B and (doubly) at the points of members of $\xi = \zeta \otimes \eta$ is the adjugate system of a net if and only if $\epsilon(\zeta) = 0$. Each net with discriminant locus Δ arises in this way for some B and ζ.*

The question of which B are permissible was discussed to some extent in § 2. It seems clear that B must be contained in the set of base points of the adjoint system of Δ; when Δ is reducible, there are also some parity conditions.

As we need the generalized jacobians, we pause to survey their basic properties, which are not easily to be found in the literature. For such as are not contained in Rosenlicht's original paper, I am indebted to Peter Newstead, who has also supplied proofs.

Any complete algebraic curve Γ has a generalized jacobian $J(\Gamma)$, which is a commutative algebraic group. $J(\Gamma)$ is complete (hence, an abelian variety) if and only if Γ is non-singular. In general, if $h\colon \bar{\Gamma} \to \Gamma$ is the normalization, there is an induced epimorphism $h^*\colon J(\Gamma) \to J(\bar{\Gamma})$. The kernel of h^* is a linear algebraic group, the quotient by whose nilradical is an algebraic torus. More precisely, $\ker h^* = \Pi_{\mathrm{P}} K(\Gamma, \mathrm{P})$ is a product of algebraic groups corresponding to the singular points P of Γ. Both globally and locally, this kernel can be expressed as a quotient of unit groups in the associated rings $\mathfrak{O}(\bar{\Gamma})^\times/\mathfrak{O}(\Gamma)^\times$. For example, at a point where r branches meet transversely, we have $(\mathbb{C}^\times)^r/(\mathbb{C}^\times) \cong (\mathbb{C}^\times)^{r-1}$; whereas at a singular point P with only one branch we have a unipotent group whose dimension is the number of positive integers not belonging to the semigroup of P, i.e. half the Milnor number of P, $\mu(\mathrm{P})$. On deforming Γ (as a plane curve, for example), the dimension of $J(\Gamma)$ remains constant, but its structure may change.

If Γ has $r(\mathrm{P})$ branches at P, the reductive quotient of $K(\Gamma, \mathrm{P})$ has dimension $(r(\mathrm{P})-1)$; $K(\Gamma, \mathrm{P})$ itself has dimension $\frac{1}{2}(\mu(\mathrm{P})+r(\mathrm{P})-1)$. These extensions all split if we merely consider the groups as topological groups; thus $K(\Gamma,\mathrm{P}) \cong (\mathbb{C}^+)^{\frac{1}{2}(\mu(\mathrm{P})-r(\mathrm{P})+1)} \times (\mathbb{C}^\times)^{r(\mathrm{P})-1} \cong \mathbb{R}^{\mu(\mathrm{P})} \times (S^1)^{r(\mathrm{P})-1}$. Notice that the calculation of these dimensions yields the formula of Milnor (1968) for the genus of a plane curve. The extension $1 \to \ker h^* \to J(\Gamma) \to J(\bar{\Gamma}) \to 1$ also splits as a sequence of topological groups, hence the sequence of elements of order 2, say

$$1 \to \Pi_{\mathrm{P}} K_2(\Gamma, \mathrm{P}) \to J_2(\Gamma) \to J_2(\bar{\Gamma}) \to 1,$$

is also split exact.

By the above, $K_2(\Gamma, \mathrm{P})$ has rank $r(\mathrm{P})-1$ over $\mathbb{F}_2$. We can interpret it explicitly as follows. Since

$$K(\Gamma, \mathrm{P}) \cong \mathfrak{O}(\bar{\Gamma}, \mathrm{P})^\times/\mathfrak{O}(\Gamma, \mathrm{P})^\times,$$

we must take the functions $\phi\colon h^{-1}(\mathrm{P}) \to \{\pm 1\}$ which are the elements of order 2 in $\mathfrak{O}(\bar{\Gamma}, \mathrm{P})$, and factor out the subgroup of constant functions on $h^{-1}(\mathrm{P})$. Thus elements of $K_2(\Gamma, \mathrm{P})$ can be interpreted as partitions of the set of branches of Γ at P into two disjoint subsets (namely $\phi^{-1}\{1\}$ and $\phi^{-1}\{-1\}$).

3.3. *Review of quadratic functions over* $\mathbb{F}_2$.

We are about to reconsider the lists in §2, this time seeking to understand why the numbers of nets of each type are just so. Before doing so, however, it is necessary to recall the basic facts concerning quadratic forms over $\mathbb{F}_2$.

Let E be an affine space over $\mathbb{F}_2$ with underlying vector space V, $q\colon E \to \mathbb{F}_2$ a quadratic map, associated to the symmetric bilinear map $b\colon V \times V \to \mathbb{F}_2$; i.e. for any $x, y \in V$ and $z \in E$ we have

$$q(x+y+z)+q(x+z)+q(y+z)+q(z) = b(x,y).$$

We do not assume that b is non-singular.

Write R for the radical subspace $\{x \in V\colon b(x, V) = 0\}$. Then for $x \in R$, $c(x) = q(x+z)+q(z)$ is independent of $z \in E$, and $c\colon R \to \mathbb{F}_2$ is a homomorphism. There are now two cases. If c is non-zero – say $c(x) = 1$ – then for each $z \in E$, $q(x+z)$ and $q(z)$ have distinct values. Hence q takes the values 0 and 1 equally often. In this case we define $\Phi(q) = 0$.

If, however, c vanishes we have $q(x+z) = q(z)$ for all $x \in R$, $z \in E$. We write $V' = V/R$, $E' = E/R$ for the quotients by R: E' is an affine space with vector space V'. Further, b defines a non-singular form $b'\colon V' \times V' \to \mathbb{F}_2$ and q a quadratic $q'\colon E' \to \mathbb{F}_2$ associated to b'. Choose a symplectic basis $\{e_i, f_i\colon 1 \leqslant i \leqslant k\}$ for (V', b') and a base point $z_0 \in E'$. Then $x \to q''(x) = q'(x+z_0)+q'(z_0)$ is a quadratic form on V' associated to b'; its Arf invariant is $\sum_{i=1}^{k} q''(e_i)\, q''(f_i)$ and if this is 0 (resp. 1) then q'' takes the value 0 (resp. 1) for $2^{2k-1}+2^{k-1}$ elements of V'.

It is convenient to set $\Phi(q) = \Phi(q') = +1$ (resp. -1) if $\mathrm{Arf}\,(q'')+q'(z_0) = 0$ (resp. 1). Then the number of $z \in E'$ with $q'(z) = 0$ is $2^{2k-1}+\Phi(q')\,2^{k-1}$, and if $r = \dim R$ the number of $z \in E$ with $q(z) = 0$ is $2^{2k+r-1}+\Phi(q)\,2^{k+r-1}$. This is also true in the case $\Phi(q) = 0$.

For a singular curve $\Gamma = \Delta_B$ as above, we may identify $J_2(\Gamma)$ with $H^1(\Gamma, \mathbb{F}_2)$. Clearly $K_2(\Gamma)$ is the radical of the bilinear form defined by cup product, so the corresponding V' is $H^1(\bar{\Gamma}, \mathbb{F}_2)$ and the parameter k is the genus p of Γ (or of Δ). The parameter $r = \Sigma\{r(\mathrm{P})-1 : \mathrm{P}$ a singular point of $\Gamma\}$ is easily written down in each case. Thus a first check on the above conjecture is that, in each case, the number of nets obtained equals one of the integers $2^{2p+r-1}+\Phi 2^{p+r-1}$ with $\Phi = 1, 0$ or -1. We now go through the cases, and determine the values of Φ.

3.4. *Tables and conjectures for Δ irreducible*

We now verify that for the cases considered in (2.3) the number of nets equals one of the values above. In table 7 below we recapitulate the data from tables 1–3, and indicate in each case the genus p, and tabulate (r, Φ) where r denotes the number of nodes with branches not separated by blowing up and Φ is determined so that the number of nets equals $2^{p+r-1}(2^p+\Phi)$.

Table 7

singularities	p	base points						
none	3	none	(0, +1)					
L		—	L	L, L	L, L, L			
A_1	2	(1, 0)	(0, +1)					
A_2	2	(0, −1)	(0, +1)					
A_3	1	(1, −1)	(1, 0)	(0, +1)				
A_4	1	(0, −1)	(0, −1)	(0, +1)				
A_5	0	(1, 0)	(1, −1)	(1, 0)	(0, +1)			
A_6	0	(0, +1)	(0, −1)	(0, −1)	(0, +1)			
L+M		—	L	L, L	M	L, M	L, L, M		
A_1+A_1	1	(2, 0)	(1, 0)					
A_2+A_1	1	(1, 0)	(1, 0)					
A_2+A_2	1	(0, +1)	(0, −1)					
A_3+A_1	0	(2, 0)	(2, 0)	(1, 0)	(1, −1)	(1, 0)	(0, +1)		
A_3+A_2	0	(1, +1)	(1, 0)	(0, −1)	(1, −1)	(1, 0)	(0, +1)		
A_4+A_1	0	(1, 0)	(1, 0)	(1, 0)	(0, −1)	(0, −1)	(0, +1)		
A_4+A_2	0	(0, +1)	(0, +1)	(0, −1)	(0, −1)	(0, −1)	(0, +1)		
L+M+N		—	L	M	N	L, M	L, N	M, N	L, M, N
$A_1+A_1+A_1$	0	(3, 0)	(2, 0)	(2, 0)	(2, 0)	(1, 0)	(1, 0)	(1, 0)	(0, +1)
$A_2+A_1+A_1$	0	(2, 0)	(2, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(0, −1)	(0, +1)
$A_2+A_2+A_1$	0	(1, 0)	(1, 0)	(1, 0)	(0, +1)	(1, 0)	(0, −1)	(0, −1)	(0, +1)
$A_2+A_2+A_2$	0	(0, −1)	(0, +1)	(0, +1)	(0, +1)	(0, −1)	(0, −1)	(0, −1)	(0, +1)

We observe that when ordinary singularities (A_1 and A_2) alone are present, we have $\Phi = 0$ if and only if $\Gamma = \Delta_B$ has a node; otherwise $\Phi = (-1)^{\kappa}$, where κ is the number of cusps on Γ. More generally, the value of Φ in all cases is consistent with the following.

Conjecture 4. (interim) *The value of Φ depends only on the singularities of the curve $\Gamma = \Delta_B$. Moreover, to each type of singularity, we can assign a number $\Phi^{\mathrm{P}} = -1$, 0 or $+1$ and then $\Phi(\Gamma) = \Pi\{\Phi^{\mathrm{P}} : \mathrm{P}$ singular on $\Gamma\}$.*

In particular, for type A_1, A_2, A_3, A_4, A_5 and A_6 we have

$$\Phi = 0, -1, -1, -1, 0 \quad \text{and} \quad +1.$$

Familiarity with patterns of Arf invariants and Clifford algebras now suggests that $\Phi = 0$ for A_{4k+1}, whereas for A_{4k+r} $(r = 2, 3, 4)$ we have $\Phi = (-1)^{k+1}$.

Before proceeding to reducible curves and curves with triple points, we seek to interpret this conjecture in a more intrinsic way. Recall that $\Phi = 0$ if and only if c is non-zero on the radical $K_2(\Gamma) = \Pi_P K_2(\Gamma, P)$. The simplest way to achieve the above multiplicative property is then if $c|K_2(\Gamma, P)$ can be given an intrinsic local description.

Now the double points with two branches are the A_{2k+1}, and we suggested above that $\Phi = 0$, or equivalently $c \neq 0$ for these precisely when k is even. This is equivalent to the intersection of the two branches being odd. This suggests the following, much more explicit conjecture:

Conjecture 4 (i). *For* P *a singular point of* Γ *with several branches, and* $\phi \in K_2(\Gamma, P)$ *corresponding to the partition of these branches into two sets, with unions* β *and* β', $c(\phi)$ *equals the local intersection number* $\beta\beta'$ (mod 2).

Let us verify that this is indeed a homomorphism $K_2(\Gamma, P) \to \mathbb{F}_2$. If $\phi_3 = \phi_1 + \phi_2$ in $K_2(\Gamma, P)$, we can partition the branches at P into four sets (some of which may be empty) with unions $\beta_0, \beta_1, \beta_2$ and β_3 so that $\phi_1 = (\beta_0 \cup \beta_1, \beta_2 \cup \beta_3)$, $\phi_2 = (\beta_0 \cup \beta_2, \beta_1 \cup \beta_3)$ and hence $\phi_3 = (\beta_0 \cup \beta_3, \beta_1 \cup \beta_2)$ Then

$$\begin{aligned} c(\phi_1) + c(\phi_2) &\equiv (\beta_0 + \beta_1)(\beta_2 + \beta_3) + (\beta_0 + \beta_2)(\beta_1 + \beta_3) \\ &\equiv \beta_0\beta_2 + \beta_0\beta_3 + \beta_1\beta_2 + \beta_1\beta_3 + \beta_0\beta_1 + \beta_0\beta_3 + \beta_1\beta_2 + \beta_2\beta_3 \\ &\equiv \beta_0\beta_1 + \beta_0\beta_2 + \beta_1\beta_3 + \beta_2\beta_3 \\ &\equiv (\beta_0 + \beta_3)(\beta_1 + \beta_2) = c(\phi_3) \pmod 2. \end{aligned}$$

We turn to the assertion of the conjecture concerning non-zero values of Φ. Again, it is natural to expect much more to be true. We propose:

Conjecture 4 (ii). *If* c *vanishes on* $K_2(\Gamma)$, *the non-singular form associated to* q *is the Riemann form on* $J_2(\bar{\Gamma})$. *The constant value of* q *on* $K_2(\Gamma)$ *is the sum of terms* Φ_P *depending only on the types of singular points* P *of* Γ.

The above conjectures, along with $\Phi(A_{4k+r}) = (-1)^{k+1}$ for $r = 2, 3, 4$, may be considered as the main conclusions of this paper. For completeness, we finally discuss the modifications necessary for reducible curves.

3.5. *The case of reducible curves*

Here we must be careful of the possibilities that Γ is disconnected; and certainly $\bar{\Gamma}$ is so. The most convenient terminology here is that of cohomology. The relative cohomology groups of $h: \bar{\Gamma} \to \Gamma$ reduce, by excision, to sums of groups corresponding to the singular points P. These in turn are the cokernels $\tilde{H}^0(h^{-1}(P); \mathbb{F}_2)$ of the natural injections $H^0(P; \mathbb{F}_2) \to H^0(h^{-1}(P); \mathbb{F}_2)$; in the irreducible case, this was the group denoted $K_2(\Gamma, P)$ and corresponds to the partitions of branches at P into two sets. We now have the exact sequence

$$0 \to H^0(\Gamma; \mathbb{F}_2) \xrightarrow{h^*} H^0(\bar{\Gamma}; \mathbb{F}_2) \to \oplus_P \tilde{H}^0(h^{-1}(P); \mathbb{F}_2) \to H^1(\Gamma; \mathbb{F}_2) \xrightarrow{h^*} H^1(\bar{\Gamma}; \mathbb{F}_2) \to 0.$$

The radical of the cup product form is still the kernel of h^*, so a quadratic form on $H^1(\Gamma; \mathbb{F}_2)$ (or a principal homogeneous space of this) still induces a homomorphism c of ker h^* into $\mathbb{F}_2$. It is natural to extend conjecture 4(i) to demand that the induced homomorphism of $\oplus_P \tilde{H}^0(h^{-1}(P); \mathbb{F}_2)$ has the values there prescribed. This imposes a consistency condition, that this homomorphism does indeed factorize through ker h^*, or equivalently, that it vanish on the image of $H^0(\bar{\Gamma}; \mathbb{F}_2)$.

Now $H^0(\bar{\Gamma}; \mathbb{F}_2)$ is generated by the classes η_i which are 1 one one component $\bar{\Gamma}_i$ of $\bar{\Gamma}$ and 0 on the others. The image of η_i in $H^0(h^{-1}(P); \mathbb{F}_2)$ corresponds to the selection of those branches at P which belong to $\bar{\Gamma}_i$. Thus $c_P(\eta_i)$ is the local intersection number at P of Γ_i with $(\Gamma - \Gamma_i)$, taken modulo 2; and $c(\eta_i)$ is the global intersection number of Γ_i with $(\Gamma - \Gamma_i)$, mod 2. Hence

Proposition. *Conjecture* 4(i) *implies the consistency condition, that for each irreducible component* Γ_i *of* Γ, *the intersection number* $\Gamma_i(\Gamma - \Gamma_i)$ *is even.*

Observe that applying this condition to the blown-up curves Δ_B provides precisely the parity condition noted earlier as necessary for the set B. This may be adduced as further evidence in support of Conjecture 4(i): notice that this argument is in no way restricted to the case $n = 4$.

To check, we reconsider the numerical data. Recall the notation: we have the quartic Δ with adjugate base points B; blowing these up yields $\Delta_B = \Gamma$, with normalization $\bar{\Delta} = \bar{\Gamma}$. Write p for the sum of the genera of components of any of these; r is the rank of $\ker h^*: H^1(\Gamma; \mathbb{F}_2) \to H^1(\bar{\Gamma}; \mathbb{F}_2)$. When Γ only has double points, r equals the number of those with two branches, minus the number of irreducible components of Γ, plus the number of connected components.

First we list the cases when there are just 2 components (table 8). There are 54 possibilities for (Δ, B) listed in §2, but these yield only 17 cases for Γ. We verify that the number N of nets depends only on Γ (though the multiplicity m does not: the first row of the table arises with multiplicities 8, 12, 20, 30 and 35, and the second with 4, 6, 8, 12, 15 and 16!). We may write $\Gamma = \Gamma_1 \cup \Gamma_2$ with Γ_2 of genus 0 and non-singular; in table 8, p denotes the genus of Γ_1, $S(\Gamma_1)$ its singularities and $\Gamma_1 \cap \Gamma_2$ the singularities arising from intersections of Γ_1 with Γ_2.

In the cases when there are more than two components (and $n = 4$), each has genus 0. There are 10 further cases, represented by the diagrams in table 9 (intersections not shown do not exist). In each case we give the triple of values of (r, N, Φ).

Table 8

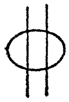
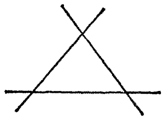
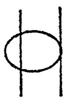
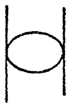
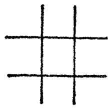
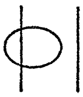
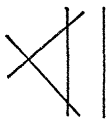
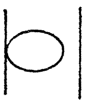
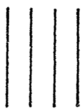
$S(\Gamma_1)$	$\Gamma_1 \cap \Gamma_2$	p	r	N	Φ
—	—	0	0	1	+1
—	$2A_1$	0	1	1	0
—	A_3	0	0	0	−1
—	$4A_1$	0	3	4	0
—	A_3+2A_1	0	2	2	0
—	$2A_3$	0	1	2	+1
—	A_5+A_1	0	1	1	0
—	A_7	0	0	1	+1
—	—	1	0	3	+1
—	$2A_1$	1	1	4	0
—	A_3	1	0	1	−1
A_1	—	0	1	1	0
A_1	$2A_1$	0	2	2	0
A_1	A_3	0	1	1	0
A_2	—	0	0	0	−1
A_2	$2A_1$	0	1	1	0
A_2	A_3	0	0	1	+1

Thus the conjectures already enunciated are seen to give correct results in all these cases: note the confirmation that A_7 corresponds to $\Phi = +1$.

I shall not give the discussion for triple points, which are subtler; the system B needs to be replaced by an ideal, and it is by no means clear (except for D_4 types) which ideals are permissible.

We can make one final remark, however: to each Δ we have associated a list of permissible B and to each B a multiplicity m_B and a group of order 2^{2p+r_B}. It is easily verified that $\Sigma_B 2^{2p+r_B} m_B = 64$ in all cases. We have shown that 36 of these correspond to nets; the remaining 28 correspond to bitangents: generalizing the result described in the non-singular case by Hesse (1855*b*).

TABLE 9

Γ	p	r	N	Φ	Γ	p	r	N	Φ
	0	2	2	0		0	1	1	0
	0	1	1	0		0	0	1	+1
	0	0	1	+1		0	1	1	0
	0	1	1	0		0	1	1	0
	0	0	0	−1		0	0	1	+1

3.6. *Concluding discussion*

It is now time to discuss the status of the conjectures in this paper. We have presented these as arising out of the experimental data in § 1.

Indeed, it would be straightforward (though tedious) to use these direct geometrical arguments to verify the conjectures in many of these special cases. For example, in the case of nets of conics we have already observed the relation to points of order 2 on the Jacobian curve of the cubic. If Δ is a quartic with a node at L, there is a similar interpretation in terms of the 6 tangents from L to Δ which relates lemma 2.1 directly to the conjectures.

There is also a more theoretical approach to this subject given in Turin (1975) and Barth (1977). Turin's paper is largely concerned with moduli, but also establishes a classification of (non-singular) nets by the curve $\Delta = 0$, a sheaf $\mathfrak{O}_S$ of rings over Δ and an $\mathfrak{O}_S$-sheaf θ (the theta-characteristic) satisfying $h^0(\theta) = 0$. This almost establishes our conjecture 3, but there are difficulties in translation: e.g. what are the precise conditions on θ?, which still leaves some questions unanswered.

Barth's paper relates the classification of 2-plane bundles over $P_2\mathbb{C}$ to nets which are non-singular, have no subpencil with a common vertex, and have rank 2 where the latter condition is defined as follows. If the net is defined by symmetric matrices A_0, A_1 and A_2 with A_1 non-singular,

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then the rank of the skew-symmetric matrix $A_0 A_1^{-1} A_2 - A_2 A_1^{-1} A_0$ depends only on the net, and may be called the rank of the net. This invariant does not appear to have a convenient interpretation in terms of our approach to classification (example: for nets of conics, type E and its specializations G, H, I, I* have rank 0; the rest – it is enough to check type G* – have rank 2).

My feeling is that it should now be feasible to prove conjectures 1(i), 3 and 4 by these methods. The questions concerning multiplicity are in any case of somewhat less interest.

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